Normal Subgroups

A subgroup H of a group G is called a normal subgroup of G if $aH = Ha$ for all a in G. This is denoted by $H \triangleleft G$.

In other words, a subgroup H is normal in G if the left cosets = right cosets

A common way to visualize Normal Subgroups is through the use of Cayley tables. We will do this through the help of the multiplication tables provided by Group Explorer.

Z_{12}

Run Group Explorer and open the group Z_{12}
Before opening the multiplication table, scroll down and read through the group's properties: you will see that Z_{12} is abelian, cyclic, and has six subgroups
Now click on the multiplication table to open it and maximize the window
On the right is the list of subsets as well as a Subsets tab and a Table tab
Switch to the Table tab
From the dropdown list titled "Organize by subgroup:," choose H_1 from the list
Now that the "separate cosets by:" is available, move the bar all the way to the right for a maximum separation
Notice the symmetry between cosets along the left column and those along the top row
This symmetry is what tells us that the subgroup is Normal

S_3

Open S_3 from the G.E. Library
Notice some of the properties of S_3: it is not Abelian and has six subgroups
Once again open the Multiplication table and maximize the window
On the right, go over the list of subgroups before switching to the Table tab

1. How many subgroups of S_3 are Normal?
2. What subgroup(s) is Normal?

D_4

Open D_4 from the G.E. Library
Once again, you can see from the list of properties that D_4 is not Abelian
Click on the multiplication table to open it and maximize the window
Take a look at the list of subgroups on the right

1. How many of these subgroups do you think are normal?

Switch to the Table tab

2. List all Normal subgroups of D_4

In the Table tab, choose $H_2 = \{e, r^2\}$ from the dropdown menu
Separate the cosets by the maximum amount

3. Regarding each coset as an element, what could this table represent?

Q_4

Choose Q_4 from the G.E. Library
Again, we can tell from the properties that this group is not Abelian
Bring up the multiplication table by clicking on it and then maximize the window
Switch immediately to the Table view and begin going through the subgroups

1. How many of these subgroups are Normal?

Look at the subgroup H1 separated into its cosets

2. Regarding again each coset as an element, what could this table represent?
3. In general, what do the multiplication tables of normal subgroups show?

Generalizations

Recall the index of a subgroup [G:H] = |G| / |H|

1. For the groups we looked at, what can you say about H when [G:H] = 2?
2. Is this always the case? Try to prove or disprove this.

3. Why have Abelian groups been excluded from these exercises? (Hint: take a look at a few Abelian groups, what can you say about their subgroups?)