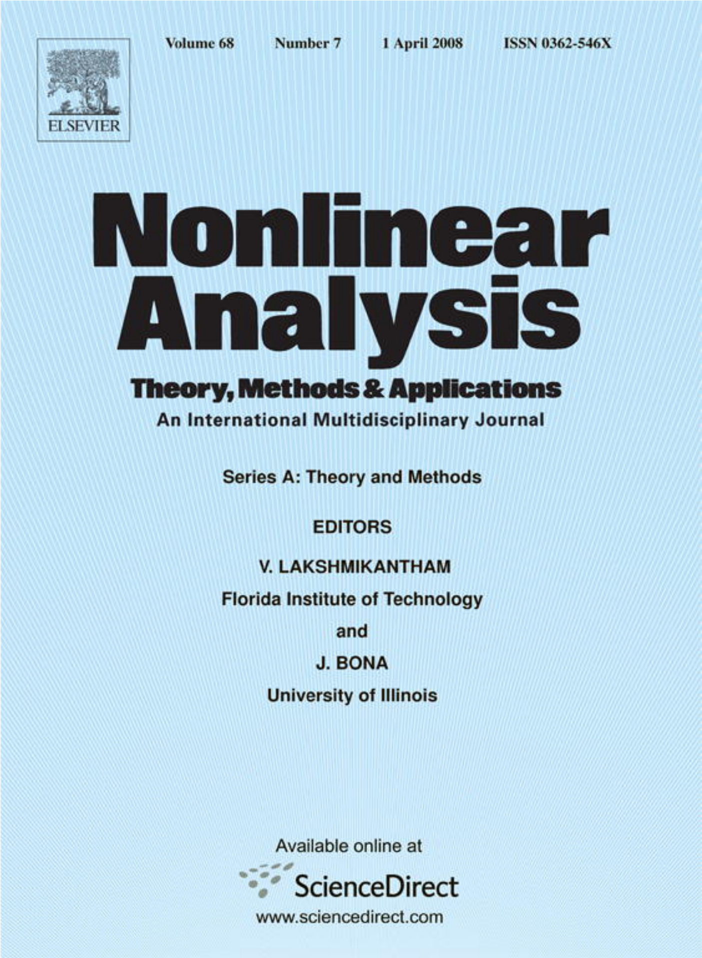

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Global and blow-up solutions for a mutualistic model

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Abstract

We study the global and blow-up solutions for a strong degenerate reaction–diffusion system modeling the interactions of two biological species. The local existence and uniqueness of a classical solution are established. We further give the critical exponent for reaction and absorption terms for the existence of global and blow-up solutions. We show that the solution may blow up if the intraspecific competition is weak. This supports ecologist A.J. Nicholson's conclusion that intraspecific competition is the main factor regulating population size.

Keywords: Mutualistic model; Degenerate reaction–diffusion system; Global solution; Blow-up solution

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1. Introduction

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following nonlinear parabolic system:

$$\begin{cases} u_t = u^p \Delta u + u^{p+1}(a_1 - b_1 u^r + c_1 v^l) & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = v^q \Delta v + v^{q+1}(a_2 + b_2 u^s - c_2 v^m) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

where $p, q, r, s \geq 1, l \geq 1, m, a_i, b_i, c_i$ $(i = 1, 2)$ are positive constants. The initial data $u_0(x)$ and $v_0(x)$ satisfy

$$\begin{cases} u_0(x), v_0(x) \in C^1(\bar{\Omega}), \qquad u_0(x), v_0(x) > 0 & \text{in } \Omega, \\ u_0(x) = v_0(x) = 0, \qquad \dfrac{\partial u_0}{\partial \eta} < 0, \qquad \dfrac{\partial v_0}{\partial \eta} < 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Here η is the outward normal vector on $\partial\Omega$.

We call (u, v) a classical solution of (1) if $(u, v) \in [C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))]^2$ for some $0 < T \leq \infty$ and (u, v) satisfies the differential equations in (1) and the initial and boundary conditions.

System (1) is usually referred as the cooperative two-species Lotka–Volterra model. It provides a simple model for describing the interaction of two diffusive biological species. The unknown functions u and v represent the densities

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of two species. a_1 and a_2 are the growth rates. The reaction terms v^l and u^s represent the assumption that each species finds its subsistence from the activity of the other one, or interspecific competition. The absorption terms u^r and v^m represent the competition among the same species, or intraspecific competition.

In [7], Pao studied the following mutualistic model:

$$\begin{cases} u_t = d_1 \Delta u + u(a_1 - b_1 u + c_1 v) & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = d_2 \Delta v + v(a_2 + b_2 u - c_2 v) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) & \text{for } x \in \Omega. \end{cases} \tag{3}$$

He showed that the solution of (3) is unique and global when $b_2 c_1 < b_1 c_2$; the solution blows up for any $a_1 \geq 0, a_2 \geq 0$ with suitable initial data when $b_2 c_1 > b_1 c_2$. For the critical case $b_2 c_1 = b_1 c_2$, he showed that the solution blows up in finite time for large a_1 and a_2. These results imply that the solution is global if the intraspecific competition is strong, while the solution may blow up if the intraspecific competition is weak.

For the following similar system:

$$\begin{cases} u_t = u^p(\Delta u + av) \\ v_t = v^q(\Delta v + bu) \end{cases} \tag{4}$$

Wang [8] proved that the solution (u, v) exists globally if and only if $ab \leq \lambda_1^2$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition. For similar systems that have been studied, we refer the readers to [2–6,9,10].

When $p = q, a_1 = a_2, b_i = c_i = 0$ and $u_0(x) = v_0(x)$, system (1) is then reduced to a single initial boundary value problem:

$$\begin{cases} u_t = u^p(\Delta u + a_1 u) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \end{cases} \tag{5}$$

This problem has been discussed by many authors. For example, for when $p = 2$, Friedman and Mcleod [2] proved that if $\lambda_1 > a_1$ then the solution exists globally but it blows up in finite time for $\lambda > a_1$. For other related results, see for example [10] and references therein. For a survey of blowup phenomena in parabolic equations, we refer the readers to [1].

A key feature of (1) is its degeneracy, since $u = v = 0$ on $\partial\Omega$. Our main goal in this paper is to establish the local existence as well as the global existence and nonexistence of the solutions. Our main results are stated in the following theorems.

Theorem 1. *If $ls < rm$, then all solutions of* (1) *are global and uniformly bounded.*

Theorem 2. *If $ls = rm$, then:*

(1) (1) *has a unique global solution (u, v) which is uniformly bounded for $b_1^s c_2^r > b_2^r c_1^s$, i.e., $b_1^m c_2^l > b_2^l c_2^m$.*
(2) *The solution of* (1) *blows up in finite time for $b_1^s c_2^r \leq b_2^r c_1^s$ provided that $\min\{a_1, a_2\} > \lambda_1$.*

Theorem 3. *If $ls > rm$, then the solution of* (1) *blows up in finite time for $b_1^s c_2^r \leq b_2^r c_1^s$ provided that $\min\{a_1 - b_1, a_2\} > \lambda_1$ or $\min\{a_1, a_2 - b_2\} > \lambda_1$.*

This paper is organized as follows. In the next section, we establish the local existence, uniqueness and give several comparison principles. In Section 3, we prove Theorems 1–3.

2. Local existence, uniqueness and comparison principle

2.1. Local existence

Since $u = v = 0$ on the boundary $\partial\Omega$, the equations in (1) are not strictly parabolic. The standard parabolic theory cannot be used to prove the existence of a solution directly. To prove local existence, we modify the boundary conditions. We consider the following regularized system:

$$\begin{cases} u_{\epsilon t} = f_\epsilon(u_\epsilon)\Delta u_\epsilon + f_\epsilon(u_\epsilon)u_\epsilon(a_1 - b_1 u_\epsilon^r + c_1 v_\epsilon^l) & \text{in } \Omega \times \mathbb{R}^+, \\ v_{\epsilon t} = g_\epsilon(v_\epsilon)\Delta v_\epsilon + g_\epsilon(v_\epsilon)v_\epsilon(a_2 + b_2 u_\epsilon^s - c_c v_\epsilon^m) & \text{in } \Omega \times \mathbb{R}^+, \\ u_\epsilon(x,t) = v_\epsilon(x,t) = \epsilon & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_\epsilon(x,0) = u_0(x) + \epsilon, \qquad v_\epsilon(x,0) = v_0(x) + \epsilon & \text{for } x \in \bar{\Omega}, \end{cases} \tag{6}$$

where f_ϵ and g_ϵ are smooth and positive functions satisfying

$$f_\epsilon(u) = \begin{cases} u^p, & u \geq \epsilon, \\ \epsilon^p, & u < \epsilon, \end{cases} \quad \text{and} \quad g_\epsilon(v) = \begin{cases} v^q, & v \geq \epsilon, \\ \epsilon^q, & v < \epsilon. \end{cases} \tag{7}$$

By standard parabolic theory, we get the following lemma:

Lemma 4. *There exists a unique solution* $u_\epsilon, v_\epsilon \in C(\bar{\Omega} \times [0, T(\epsilon))) \cap C^{2,1}(\Omega \times (0, T(\epsilon)))$ *for system* (6) *such that* $u_\epsilon \geq 0$ *and* $v_\epsilon \geq 0$, *where* $0 < T(\epsilon) \leq \infty$.

The following proposition provides a lower bound of the solution for system (6).

Proposition 5. *If* $\epsilon \leq \min\{(a_1/b_1)^{1/r}, (a_2/c_2)^{1/m}\}$, *then* $u_\epsilon \geq \epsilon$ *and* $v_\epsilon \geq \epsilon$ *in* $\bar{\Omega} \times [0, T(\epsilon))$.

To prove this proposition, we need the following lemmas.

Lemma 6. *Let* $z_i \in C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))$ $(i = 1, 2)$ *and satisfy*

$$\begin{cases} z_{it} - a_i(x,t)\Delta z_i \geq \displaystyle\sum_{j=1}^{2} b_{ij}(x,t) z_j & \textit{in } \Omega \times (0, T), \\ z_i(x,t) \geq 0 & \textit{on } \partial\Omega \times (0, T), \\ z_i(x,t) \geq 0 & \textit{for } x \in \bar{\Omega}. \end{cases} \tag{8}$$

If $a_i, b_{ij} \in C(\bar{\Omega} \times [0, T))$ *such that* $a_i > 0$ *and* $b_{ij} \geq 0$ *for* $i \neq j$ *in* $\Omega \times (0, T)$, *then* $z_i \geq 0$ *in* $\bar{\Omega} \times [0, T)$.

Lemma 7. *Let* $z_i \in C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))$ $(i = 1, 2)$ *and satisfy*

$$\begin{cases} z_{it} - a_i(x,t)\Delta z_i = P_i(z_1, z_2) & \textit{in } \Omega \times (0, T), \\ z_i(x,t) \geq 0 & \textit{on } \partial\Omega \times (0, T), \\ z_i(x,t) \geq 0 & \textit{for } x \in \bar{\Omega}. \end{cases} \tag{9}$$

If $a_i \in C(\bar{\Omega} \times [0, T))$ *such that* $a_i > 0$ *and* $P_i(z_1, z_2)$ *is Lipschitz continuous with respect to* z_i *and it holds that* $P_i(z_1, z_2) \geq 0$ *for* $(z_1, z_2) \in \mathbb{R}_+^2$ *whenever* $z_i = 0$, *then* $z_i \geq 0$ *for all* $x \in \bar{\Omega}$.

The proof of Lemma 6 can be found in [7]. Here we prove Lemma 7.

Proof. We introduce the following notation:

$$\mathbf{Z} = (z_1, z_2), \qquad \mathbf{Z}_1 = (0, z_2), \qquad \mathbf{Z}_2 = (z_1, 0).$$

Observe that $P_i(\mathbf{Z})$ can be rewritten as

$$P_i(\mathbf{Z}) = P_i(\mathbf{Z}) - P_i(\mathbf{Z}_i) + P_i(\mathbf{Z}_i) \geq P_i(\mathbf{Z}) - P_i(\mathbf{Z}_i)$$

since $P_i(\mathbf{Z}_i) \geq 0$.

By the Lipschitz continuity of P_i with respect to z_i, there exists $0 \leq \bar{z}_i \leq z_i$ such that

$$P_i(\mathbf{Z}) - P_i(\mathbf{Z}_i) = \left(\frac{\partial P_i}{\partial Z_i}(\mathbf{Z}) \bigg|_{z_i = \bar{z}_i} \right) z_i.$$

Moreover, the Lipschitz continuity guarantees that for $t \in [0, T)$, there exists a positive constant C such that

$$\left| \frac{\partial P_i}{\partial Z_i}(\mathbf{Z}) \bigg|_{z_i = \bar{z}_i} \right| \leq C.$$

It follows that

$$P_i(\mathbf{Z}) \geq -Cz_i \quad \text{for } t \in [0, T).$$

Applying Lemma 6, we have $z_i(x,t) \geq 0$ in $\bar{\Omega} \times [0, T)$. □

We are now in position to prove Proposition 5.

Proof. Let $z_1 = u_\epsilon - \epsilon$, $z_2 = v_\epsilon - \epsilon$; we have

$$\begin{cases} z_{1t} = \tilde{f}_\epsilon(z_1)\left[\Delta z_1 + (a_1 - b_1(z_1+\epsilon)^r)z_1 + \epsilon\left(a_1 - b_1(z_1+\epsilon)^r\right)\right] + c_1\tilde{f}_\epsilon(z_1)(z_1+\epsilon)(z_2+\epsilon)^l \\ \quad \text{in } \Omega \times (0, T(\epsilon)), \\ z_{2t} = \tilde{g}_\epsilon(z_2)\left[\Delta z_2 + (a_2 - c_2(z_2+\epsilon)^m)z_2 + \epsilon\left(a_2 - c_2(z_2+\epsilon)^m\right)\right] + b_2\tilde{g}_\epsilon(z_2)(z_2+\epsilon)(z_1+\epsilon)^l \\ \quad \text{in } \Omega \times (0, T(\epsilon)), \\ z_1(x,t) = z_2(x,t) = 0 \quad \text{on } \partial\Omega \times (0, T(\epsilon)), \\ z_1(x,0) = u_0(x), \qquad z_2(x,0) = v_0(x) \quad \text{for } x \in \bar{\Omega}, \end{cases} \tag{10}$$

where

$$\tilde{f}_\epsilon(z_1) = \begin{cases} (z_1+\epsilon)^p, & z_1 \geq 0, \\ \epsilon^p, & z_1 < 0, \end{cases} \quad \text{and} \quad \tilde{g}_\epsilon(z_2) = \begin{cases} (z_2+\epsilon)^q, & z_2 \geq 0, \\ \mathrm{e}^q, & z_2 < 0. \end{cases}$$

Let

$$P_1 = \tilde{f}_\epsilon(z_1)\left[(a_1 - b_1(z_1+\epsilon)^r)z_1 + \epsilon\left(a_1 - b_1(z_1+\epsilon)^r\right) + c_1(z_1+\epsilon)(z_2+\epsilon)^l\right],$$

$$P_2 = \tilde{g}_\epsilon(z_2)\left[(a_2 - c_2(z_2+\epsilon)^m)z_2 + \epsilon\left(a_2 - c_2(z_2+\epsilon)^m\right) + b_2(z_2+\epsilon)(z_1+\epsilon)^l\right].$$

We can show that P_1 and P_2 satisfy the all conditions in Lemma 7 provided that $\epsilon \leq \min\{(a_1 b_1)^{1/r}, (a_2/c_2)^{1/m}\}$. Thus $u_\epsilon \geq \epsilon$ and $v_\epsilon \geq \epsilon$ in $\bar{\Omega} \times [0, T(\epsilon))$. This ends the proof of this proposition. □

Now that we have $\epsilon \leq \min\{(a_1/b_1)^{1/r}, (a_2/c_2)^{1/m}\}$, $u_\epsilon \geq \epsilon$ and $v_\epsilon \geq \epsilon$ which gives $f_\epsilon(u_\epsilon) = u_\epsilon^p$ and $g_\epsilon(v_\epsilon) = v_\epsilon^q$ and hence (u_ϵ, v_ϵ) solves the following problem:

$$\begin{cases} u_{\epsilon t} = u_\epsilon^p \Delta u_\epsilon + u_\epsilon^{p+1}(a_1 - b_1 u_\epsilon^r + c_1 v_\epsilon^l) & \text{in } \Omega \times (0, T(\epsilon)), \\ v_{\epsilon t} = v_\epsilon^q \Delta v_\epsilon + v_\epsilon^{q+1}(a_2 + b_2 u_\epsilon^s - c_2 v_\epsilon^m) & \text{in } \Omega \times (0, T(\epsilon)), \\ u_\epsilon(x,t) = v_\epsilon(x,t) = \epsilon & \text{on } \partial\Omega \times (0, T(\epsilon)), \\ u_\epsilon(x,0) = u_0(x) + \epsilon, \qquad v_\epsilon(x,0) = v_0(x) + \epsilon & \text{for } x \in \bar{\Omega}. \end{cases} \tag{11}$$

Problem (11) is not degenerate since $u_\epsilon, v_\epsilon \geq \epsilon$ for ϵ sufficiently small. Since (11) is quasimonotone nondecreasing, we can apply the comparison principle (see chapter IV, 32 of [11]) and we have the following

Proposition 8. *Let* $\underline{u}_\epsilon, \underline{v}_\epsilon \in C(\bar{\Omega} \times [0, T(\epsilon))) \cap C^{2,1}(\Omega \times (0, T(\epsilon)))$ *and satisfy*

$$\begin{cases} \underline{u}_{\epsilon t} \leq \underline{u}_\epsilon^p \Delta \underline{u}_\epsilon + \underline{u}_\epsilon^{p+1}(a_1 - b_1 \underline{u}_\epsilon^r + c_1 \underline{v}_\epsilon^l) & \text{in } \Omega \times (0, T(\epsilon)), \\ \underline{v}_{\epsilon t} \leq \underline{v}_\epsilon^q \Delta \underline{v}_\epsilon + \underline{v}_\epsilon^{q+1}(a_2 + b_2 \underline{u}_\epsilon^s - c_2 \underline{v}_\epsilon^m) & \text{in } \Omega \times (0, T(\epsilon)), \\ \underline{u}_\epsilon(x,t) \leq \epsilon, \qquad \underline{v}_\epsilon(x,t) \leq \epsilon & \text{on } \partial\Omega \times (0, T(\epsilon)), \\ \underline{u}_\epsilon(x,0) \leq u_0(x) + \epsilon, \qquad \underline{v}_\epsilon(x,0) \leq v_0(x) + \epsilon & \text{for } x \in \bar{\Omega}. \end{cases}$$

Then $(\underline{u}_\epsilon, \underline{v}_\epsilon) \leq (u_\epsilon, v_\epsilon)$ *on* $\bar{\Omega} \times [0, T(\epsilon))$.

Next we shall establish the following proposition:

Proposition 9. *If* $\epsilon_1 < \epsilon_2 \leq \min\{(a_1/b_1)^{1/r}, (a_2/c_2)^{1/m}\}$, *then* $T(\epsilon_1) \geq T(\epsilon_2)$ *and* $(u_{\epsilon_1}, v_{\epsilon_1}) \leq (u_{\epsilon_2}, v_{\epsilon_2})$ *on* $\bar{\Omega} \times [0, T(\epsilon_2))$.

Proof. To prove this proposition, we let $W_1 = u_{\epsilon_2} - u_{\epsilon_1}$ and $W_2 = v_{\epsilon_2} - v_{\epsilon_1}$. From (11), we have

$$\begin{cases} W_{1t} = u_{\epsilon_2}^p \Delta W_1 + b_{11}W_1 + b_{12}W_2 & \text{in } \Omega \times (0, \min\{T(\epsilon_1), T(\epsilon_2)\}), \\ W_{2t} = v_{\epsilon_2}^q \Delta W_2 + b_{21}W_1 + b_{22}W_2 & \text{in } \Omega \times (0, \min\{T(\epsilon_1), T(\epsilon_2)\}), \\ W_1(x,t) = W_2(x,t) = \epsilon_2 - \epsilon_1 > 0 & \text{on } \partial\Omega \times (0, \min\{T(\epsilon_1), T(\epsilon_2)\}), \\ W_1(x,0) = W_2(x,0) = \epsilon_2 - \epsilon_1 > 0 & \text{for } x \in \bar{\Omega}, \end{cases}$$

where

$$
\begin{aligned}
b_{11} \;=\; & \Delta u_{\epsilon_1} p \int_0^1 \left[u_{\epsilon_1} + s(u_{\epsilon_2} - u_{\epsilon_1})\right]^{p-1} \mathrm{d}s + (a_1 + c_1 v_{\epsilon_1}^l)(p+1) \int_0^1 \left[u_{\epsilon_1} + s(u_{\epsilon_2} - u_{\epsilon_1})\right]^{p} \mathrm{d}s \\
& - b_1(p+r+1) \int_0^1 \left[u_{\epsilon_1} + s(u_{\epsilon_2} - u_{\epsilon_1})\right]^{p+r} \mathrm{d}s, \\
b_{12} = {} & c_1 u_{\epsilon_2}^{p+1} l \int_0^1 \left[v_{\epsilon_1} + s(v_{\epsilon_2} - v_{\epsilon_1})\right]^{l-1} \mathrm{d}s \geq 0, \\
b_{21} = {} & b_2 v_{\epsilon_2}^{q+1} s \int_0^1 \left[u_{\epsilon_1} + s(u_{\epsilon_2} - u_{\epsilon_1})\right]^{s-1} \mathrm{d}s \geq 0, \\
b_{22} \;=\; & \Delta v_{\epsilon_1} q \int_0^1 \left[v_{\epsilon_1} + s(v_{\epsilon_2} - v_{\epsilon_1})\right]^{q-1} \mathrm{d}s + (a_2 + b_2 u_{\epsilon_1}^s)(q+1) \int_0^1 \left[v_{\epsilon_1} + s(v_{\epsilon_2} - v_{\epsilon_1})\right]^{q} \mathrm{d}s \\
& - c_2(q+m+1) \int_0^1 \left[v_{\epsilon_1} + s(v_{\epsilon_2} - v_{\epsilon_1})\right]^{q+m} \mathrm{d}s.
\end{aligned}
$$

Note that Lemma 6 cannot provide us with the argument as we do not know whether b_{11} and b_{22} are continuous or not on $\partial\Omega$. However, this continuity condition can be weakened with stronger boundary and initial conditions. With the aid of the following lemma, the conclusion in Proposition 9 follows. □

Lemma 10. *Let* $z_i \in C(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$ $(i = 1,2)$ *and satisfy*

$$
\begin{cases}
z_{it} - a_i(x,t)\Delta z_i \geq \sum_{j=1}^{2} b_{ij}(x,t) z_j & \text{in } \Omega \times (0,T), \\
z_i(x,t) > 0 & \text{on } \partial\Omega \times (0,T), \\
z_i(x,t) > 0 & \text{for } x \in \bar{\Omega}.
\end{cases}
\tag{12}
$$

If $a_i, b_{ij} \in C(\Omega \times [0,T))$ *such that* $a_i > 0$ *and* $b_{ij} \geq 0$ *for* $i \neq j$ *in* $\Omega \times (0,T)$*, then* $z_i \geq 0$ *in* $\bar{\Omega} \times [0,T)$.

The proof of this lemma can be found in [4].

The following lemma provides us with a positive lower bound on (u_ϵ, v_ϵ) which will be applied in the proof of local existence.

Lemma 11. *Let* $\epsilon \leq \min\{(a_1/b_1)^{1/r}, (a_2/c_2)^{1/m}\}$, (u_ϵ, v_ϵ) *be the solution of* (11) *and the positive constant* $\rho \geq \max\{0, k^p(\lambda_1 - a_1 + b_1 k^r), k^q(\lambda_1 - a_2 + c_2 k^m)\}$ *for some* $k > 0$*; then*

$$
(u_\epsilon, v_\epsilon) \geq (k\Phi(x)\mathrm{e}^{-\rho t}, k\Phi(x)\mathrm{e}^{-\rho t}) \quad \text{in } \bar{\Omega} \times [0, T(\epsilon)).
$$

Here $\Phi(x)$ *is the eigenfunction corresponding to the first eigenvalue* λ_1 *of* $-\Delta$ *on* Ω *with homogeneous boundary condition. Moreover,* $\Phi(x)$ *can be normalized so that* $\max_{\bar{\Omega}} \Phi(x) = 1$, $\lambda_1 > 0$ *and* $\partial\Phi/\partial\eta < 0$ *on* $\partial\Omega$.

Proof. Set $\underline{u}_\epsilon(x,t) = k\phi\mathrm{e}^{-\rho t}$, $\underline{v}_\epsilon(x,t) = k\phi\mathrm{e}^{-\rho t}$ where $k > 0$ can be chosen so that $k\Phi(x) \leq \min\{u_0(x), v_0(x)\}$. A direct calculation yields

$$
\begin{aligned}
\underline{u}_\epsilon^p[\Delta\underline{u}_\epsilon + \underline{u}_\epsilon(a_1 - b_1\underline{u}_\epsilon^r + c_1\underline{v}_\epsilon^l)] \;=\; & (k\Phi(x)\mathrm{e}^{-\rho t})^p k\Phi(x)\mathrm{e}^{-\rho t}(-\lambda_1 + a_1 - b_1 k^r \Phi(x)^r \mathrm{e}^{-\rho r t} \\
& + c_1 k^l \Phi(x)^l \mathrm{e}^{-\rho l t}), \\
\underline{v}_\epsilon^q[\Delta\underline{v}_\epsilon + \underline{v}_\epsilon(a_2 + b_2\underline{u}_\epsilon^s - c_2\underline{v}_\epsilon^m)] \;=\; & (k\Phi(x)\mathrm{e}^{-\rho t})^q k\Phi(x)\mathrm{e}^{-\rho t}(-\lambda_1 + a_2 + b_2 k^s \Phi(x)^s \mathrm{e}^{-\rho s t} \\
& - c_2 k^m \Phi(x)^m \mathrm{e}^{-\rho m t}).
\end{aligned}
$$

Thus by taking

$$
\rho \geq \max\left\{0, k^p(\lambda_1 - a_1 + b_1 k^r), k^q(\lambda_1 - a_2 + c_2 k^m)\right\},
$$

we obtain

$$\begin{cases} \underline{u}_{\epsilon t} \le \underline{u}_\epsilon^p \Delta \underline{u}_\epsilon + \underline{u}_\epsilon^{p+1}(a_1 - b_1 \underline{u}_\epsilon^r + c_1 \underline{v}_\epsilon^l) & \text{in } \Omega \times (0, T(\epsilon)), \\ \underline{v}_{\epsilon t} \le \underline{v}_e^q \Delta \underline{v}_\epsilon + \underline{v}_\epsilon^{q+1}(a_2 + b_2 \underline{u}_\epsilon^s - c_2 \underline{v}_\epsilon^m) & \text{in } \Omega \times (0, T(e)), \\ \underline{u}_\epsilon(x,t) = \underline{v}_\epsilon(x,t) \le \epsilon & \text{on } \partial\Omega \times (0, T(\epsilon)), \\ \underline{u}_\epsilon(x,0) \le u_0(x) + \epsilon, \qquad \underline{v}_\epsilon(x,0) \le v_0(x) + \epsilon & \text{for } x \in \bar{\Omega}. \end{cases}$$

Proposition 8 implies that $(u_\epsilon, v_\epsilon) \ge (k\Phi(x)\mathrm{e}^{-\rho t}, k\Phi(x)\mathrm{e}^{-\rho t})$. ☐

It follows from Proposition 9 that there exists $T : 0 < T \le \infty$ and (u, v) which is defined on $\bar{\Omega} \times [0, T)$ such that $T(\epsilon) \nearrow T$ and $(u_\epsilon, v_\epsilon) \to (u, v)$ as $\epsilon \searrow 0$. By Lemma 11, (u, v) is positive in $\Omega \times (0, T)$. The standard local Schauder estimates imply that $(u, v) \in [C_{\mathrm{loc}}^{2+\alpha, 1+\alpha/2}(\Omega \times (0, T))]^2$ such that

$$(u_\epsilon, v_\epsilon) \to (u, v) \in [C_{\mathrm{loc}}^{2+\alpha, 1+\alpha/2}(\Omega_0 \times (t_0, t_1))]^2$$

for any $\Omega_0 \subset \Omega$ and $0 < t_0 < t_1 < T$. Therefore (u, v) satisfies (1) in $\Omega \times (0, T)$.

The continuity of (u, v) in $\Omega \times [0, T)$ follows from Lemma 11, L^p theory and the imbedding theorem. Similar to the arguments in [2], we can prove that (u, v) is continuous on $\partial\Omega \times (0, T)$. Therefore, we have the following local existence theorem.

Theorem 12. *Problem (1) admits a positive classical solution* (u, v) *where* $(u, v) \in [C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))]^2$ *and* $0 < T \le +\infty$.

2.2. Uniqueness

In this part we prove the uniqueness of the positive classical solution of (1) for $l \ge 1$ and $s \ge 1$. Suppose that (u_1, v_1) is another positive classical solution of (1) in $\Omega \times (0, T_1)$. By Proposition 8, $(u_1, v_1) \le (u_\epsilon, v_\epsilon)$ and thus $(u_1, v_1) \le (u, v)$.

Following [10], we define $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge \delta\}$ for $\delta > 0$ sufficiently small. Let $\Phi_\delta(x) > 0$ be the eigenfunction corresponding to the first eigenvalue λ_δ of $-\Delta\Phi_\delta(x) = \lambda\Phi_\delta(x)$, $x \in \Omega$, with homogeneous boundary condition. Multiplying the first equation in (1) by Φ_δ/u^p and integrating by parts, we have

$$\begin{aligned} \int_{\Omega_\delta} \xi(u)\Phi_\delta \mathrm{d}x = {} & \int_{\Omega_\delta} \xi(u_0)\Phi_\delta \mathrm{d}x - \int_0^t \int_{\partial\Omega} u \frac{\partial \Phi_\delta}{\partial \eta} \mathrm{d}s\mathrm{d}t \\ & - \lambda_\delta \int_0^t \int_{\Omega_\delta} u\Phi_\delta \mathrm{d}x\mathrm{d}t + \int_0^t \int_{\Omega_\delta} u(a_1 - b_1 u^r + c_1 v^l)\Phi_\delta \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{13}$$

where

$$\xi(u) = \begin{cases} \dfrac{u^{1-p}}{1-p} & p > 1, \\ \ln u & p = 1. \end{cases}$$

Similarly, we have

$$\begin{aligned} \int_{\Omega_\delta} \xi(u_1)\Phi_\delta \mathrm{d}x = {} & \int_{\Omega_\delta} \xi(u_0)\Phi_\delta \mathrm{d}x - \int_0^t \int_{\partial\Omega} u_1 \frac{\partial \Phi_\delta}{\partial \eta} \mathrm{d}s\mathrm{d}t \\ & - \lambda_\delta \int_0^t \int_{\Omega_\delta} u_1\Phi_\delta \mathrm{d}x\mathrm{d}t + \int_0^t \int_{\Omega_\delta} u_1(a_1 - b_1 u_1^r + c_1 v_1^l)\Phi_\delta \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{14}$$

Therefore

$$\begin{aligned} \int_{\Omega_\delta} [\xi(u) - \xi(u_1)]\Phi_\delta \mathrm{d}x = {} & -\int_0^t \int_{\partial\Omega} (u - u_1) \frac{\partial \Phi_\delta}{\partial \eta} \mathrm{d}s\mathrm{d}t - \lambda_\delta \int_0^t \int_{\Omega_\delta} (u - u_1)\Phi_\delta \mathrm{d}x\mathrm{d}t \\ & + a_1 \int_0^t \int_{\Omega_\delta} (u - u_1)\Phi_\delta \mathrm{d}x\mathrm{d}t - b_1 \int_0^t \int_{\Omega_\delta} (u^{r+1} - u_1^{r+1})\Phi_\delta \mathrm{d}x\mathrm{d}t \\ & + c_1 \int_0^t \int_{\Omega_\delta} (uv^l - u_1 v_1^l)\Phi_\delta \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{15}$$

Since $u \geq u_1$ on Ω_δ and for $l \geq 1$,

$$c_1 \int_0^t \int_{\Omega_\delta} (uv^l - u_1 v_1^l)\Phi_\delta \mathrm{d}x\mathrm{d}t = c_1 \int_0^t \int_{\Omega_\delta} (u - u_1)v^l \Phi_\delta \mathrm{d}x\mathrm{d}t + c_1 \int_0^t \int_{\Omega_\delta} u_1(v^l - v_1^l)\Phi_\delta \mathrm{d}x\mathrm{d}t$$
$$\leq M_1 \int_0^t \int_{\Omega_\delta} (u - u_1)\Phi_\delta \mathrm{d}x\mathrm{d}t + M_2 \int_0^t \int_{\Omega_\delta} (v - v_1)\Phi_\delta \mathrm{d}x\mathrm{d}t,$$

where $M_1 \geq 0$, $M_2 \geq 0$.

Thus

$$\int_{\Omega_\delta} [\xi(u) - \xi(u_1)]\Phi_\delta \mathrm{d}x \leq -\int_0^t \int_{\partial\Omega} (u - u_1)\frac{\partial \Phi_\delta}{\partial \eta}\mathrm{d}s\mathrm{d}t + (-\lambda_\delta + a_1 + M_1)\int_0^t \int_{\Omega_\delta} (u - u_1)\Phi_\delta \mathrm{d}x\mathrm{d}t$$
$$+ M_2 \int_0^t \int_{\Omega_\delta} (v - v_1)\Phi_\delta \mathrm{d}x\mathrm{d}t.$$

Similarly, we can multiply the second equation in (1) by Φ_δ / v^q and follow the process described above to obtain

$$\int_{\Omega_\delta} [\chi(v) - \chi(v_1)]\Phi_\delta \mathrm{d}x \leq -\int_0^t \int_{\partial\Omega} (v - v_1)\frac{\partial \Phi_\delta}{\partial \eta}\mathrm{d}s\mathrm{d}t + (-\lambda_\delta + a_a + M_3)\int_0^t \int_{\Omega_\delta} (v - v_1)\Phi_\delta \mathrm{d}x\mathrm{d}t$$
$$+ M_4 \int_0^t \int_{\Omega_\delta} (u - u_1)\Phi_\delta \mathrm{d}x\mathrm{d}t.$$

Here

$$\chi(v) = \begin{cases} \dfrac{v^{1-q}}{1-q} & q > 1, \\ \ln v & q = 1, \end{cases}$$

and $M_3 \geq 0$, $M_4 \geq 0$, $s \geq 1$.

On the other hand, it is easy to verify that

$$\int_{\Omega_\delta} [\xi(u) - \xi(u_1)]\Phi_\delta \mathrm{d}x \geq M^{-p} \int_{\Omega_\delta} (u - u_1)\Phi_\delta \mathrm{d}x$$

and

$$\int_{\Omega_\delta} [\chi(v) - \chi(v_1)]\Phi_\delta \mathrm{d}x \geq M^{-q} \int_{\Omega_\delta} (v - v_1)\Phi_\delta \mathrm{d}x,$$

where $M \geq \max_{\bar{\Omega} \times [0,\tau]}(u + v)$ for any $\tau < T$. It follows that

$$\int_{\Omega_\delta} [(u - u_1) + (v - v_1)]\Phi_\delta \mathrm{d}x \leq -\bar{M} \int_0^t \int_{\partial\Omega_\delta} [(u - u_1) + (v - v_1)]\frac{\partial \Phi_\delta}{\partial \eta}\mathrm{d}s\mathrm{d}t$$
$$+ M_5 \int_0^t \int_{\Omega_\delta} [(u - u_1) + (v - v_1)]\Phi_\delta \mathrm{d}x\mathrm{d}t, \tag{16}$$

where $\bar{M} = \max\{M^p, M^q\}$ and $M_5 = M^p(\lambda_\delta + a_1 + M_1 + M_4) + M^q(\lambda_\delta + a_2 + M_2 + M_3)$. We apply Gronwall's Lemma to (16) and then take the limit as $\delta \to 0$ and it follows that $(u, v) \equiv (u_1, v_1)$. Thus we have proved

Theorem 13. *If $s, l \geq 1$, then there exists a unique positive classical solution $(u, v) \in [C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))]^2$ where $0 < T \leq \infty$. Moreover, if $T < \infty$, then* $\lim_{t \to T^-} \max_{x \in \bar{\Omega}} u(x, t) = \lim_{t \to T^-} \max_{x \in \bar{\Omega}} v(x, t) = +\infty$.

2.3. *Comparison principles*

Proposition 14 (*Comparison Principle* 1). *Let* $\underline{u}, \underline{v} \in C(\bar{\Omega} \times [0, \underline{T})) \cap C^{2,1}(\Omega \times (0, \underline{T}))$ *and satisfy*

$$\begin{cases} \underline{u}_t \leq \underline{u}^p \Delta \underline{u} + \underline{u}^{p+1}(a_1 - b_1 \underline{u}^r + c_1 \underline{v}^l) & \text{in } \Omega \times (0, \underline{T}), \\ \underline{v}_t \leq \underline{v}^q \Delta \underline{v} + \underline{v}^{q+1}(a_2 + b_2 \underline{u}^s - c_2 \underline{v}^m) & \text{in } \Omega \times (0, \underline{T}), \\ \underline{u}(x,t) \leq 0, \qquad \underline{v}(x,t) \leq 0 & \text{on } \partial\Omega \times (0, \underline{T}), \\ \underline{u}(x,0) \leq u_0(x), \qquad \underline{v}(x,0) \leq v_0(x) & \text{for } x \in \bar{\Omega}, \end{cases}$$

Then $(\underline{u}, \underline{v}) \leq (u, v)$ *on* $\bar{\Omega} \times [0, \underline{T})$.

Proof. This follows directly from Proposition 8 and the uniqueness result since we have

$$(\underline{u}_\epsilon(x,t), \underline{v}_\epsilon(x,t)) \to (\underline{u}(x,t), \underline{v}(x,t)),$$
$$(u_\epsilon(x,t), v_\epsilon(x,t)) \to (u(x,t), v(x,t))$$

as $\epsilon \to 0^+$. □

Proposition 15 (*Comparison Principle* 2). *Let* $\bar{u}, \bar{v} \in C(\bar{\Omega} \times [0, \bar{T})) \cap C^{2,1}(\Omega \times (0, \bar{T}))$ *and satisfy*

$$\begin{cases} \bar{u}_t = \bar{u}^p \Delta \bar{u} + \bar{u}^{p+1}(a_1 - b_1 \bar{u}^r + c_1 \bar{v}^l) & \text{in } \Omega \times (0, \bar{T}), \\ \bar{v}_t = \bar{v}^q \Delta \bar{v} + \bar{v}^{q+1}(a_2 + b_2 \bar{u}^s - c_2 \bar{v}^m) & \text{in } \Omega \times (0, \bar{T}), \\ \bar{u}(x,t) \geq 0, \qquad \bar{v}(x,t) \geq 0 & \text{on } \partial\Omega \times (0, \bar{T}), \\ \bar{u}(x,0) \geq u_0(x), \qquad \bar{v}(x,0) \geq v_0(x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

Then $(\bar{u}, \bar{v}) \geq (u, v)$ *on* $\bar{\Omega} \times [0, \bar{T})$.

Proof. Let $(\bar{u}_\epsilon(x,t), \bar{v}_\epsilon(x,t))$ be the corresponding solutions of (11) with the boundary conditions replaced by $(\bar{u}_\epsilon(x,t), \bar{v}_\epsilon(x,t)) \geq (\epsilon, \epsilon)$ and the initial conditions replaced by $(\bar{u}_\epsilon(x,0), \bar{v}_\epsilon(x,0)) \geq (u_0(x) + \epsilon, v_0(x) + \epsilon)$. Since problem (11) is quasimonotone increasing in (u_ϵ, v_ϵ), applying the comparison principle for the parabolic system we have

$$(\bar{u}_\epsilon(x,t), \bar{v}_\epsilon(x,t)) \geq (u_\epsilon(x,t), v_\epsilon(x,t)).$$

Letting $\epsilon \to 0^+$ and by the uniqueness result we have $(\bar{u}(x,t), \bar{v}(x,t)) \geq (u(x,t), v(x,t))$. □

Proposition 16 (*Comparison Principle* 3). *Let* $\bar{u}, \bar{v} \in C(\bar{\Omega} \times [0, \bar{T})) \cap C^{2,1}(\Omega \times (0, \bar{T}))$ *and satisfy*

$$\begin{cases} \bar{u}_t \geq \bar{u}^p \Delta \bar{u} + \bar{u}^{p+1}(a_1 - b_1 \bar{u}^r + c_1 \bar{v}^l) & \text{in } \Omega \times (0, \bar{T}), \\ \bar{v}_t \geq \bar{v}^q \Delta \bar{v} + \bar{v}^{q+1}(a_2 + b_2 \bar{u}^s - c_2 \bar{v}^m) & \text{in } \Omega \times (0, \bar{T}), \\ \bar{u}(x,t) > 0, \qquad \bar{v}(x,t) > 0 & \text{on } \partial\Omega \times (0, \bar{T}), \\ \bar{u}(x,0) > u_0(x), \qquad \bar{v}(x,0) > v_0(x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

Then $(\bar{u}, \bar{v}) \geq (u, v)$ *on* $\bar{\Omega} \times [0, \bar{T})$.

Proof. Let $(z_1(x,t), z_2(x,t)) = (\bar{u}(x,t) - u(x,t), \bar{v}(x,t) - v(x,t))$. This proposition follows directly from Lemma 10. □

3. Global existence and nonexistence

Proof (*Proof of Theorem* 1). We construct a constant supersolution of (1). Let $(\bar{u}, \bar{v}) = (\eta_1, \eta_2)$ be such that $(u_0, v_0) \leq (\eta_1, \eta_2)$. To prove that (η_1, η_2) is a supersolution, it suffices to verify

$$b_1 \eta_1^r \geq a_1 + c_1 \eta_2^l, \qquad c_2 \eta_2^m \geq a_2 + b_2 \eta_1^s. \tag{17}$$

We choose (η_1, η_2) such that $a_1 \leq c_1 \eta_2^l$, $a_2 \leq b_2 \eta_1^s$. Then (17) holds if we can verify

$$b_1 \eta_1^r \geq 2 c_1 \eta_2^l, \qquad c_2 \eta_2^m \geq 2 b_2 \eta_1^s,$$

which is equivalent to

$$\left(\frac{2b_2}{c_2}\right)^{1/m} \eta_1^{s/m} \le \eta_2 \le \left(\frac{b_1}{2c_1}\right)^{1/l} \eta_1^{r/l}. \tag{18}$$

Since $ls < rm$, (18) clearly holds for suitably large η_1. By virtue of the comparison principle, we conclude that (u, v) is uniformly bounded. ☐

Proof (*Proof of Theorem* 2(1)). Like in the proof of Theorem 1, we choose (η_1, η_2) so that

$$a_1 \le \delta c_1 \eta_2^l, \qquad a_2 \le \delta b_2 \eta_1^s,$$

where $\delta > 0$ is to be determined later.

To show $(\bar{u}, \bar{v}) = (\eta_1, \eta_2)$ is a supersolution, it suffices to show that

$$b_1 \eta_1^r \ge (1+\delta) c_1 \eta_2^l, \qquad c_2 \eta_2^m \ge (1+\delta) b_2 \eta_1^s,$$

which is equivalent to

$$\left[\frac{(1+\delta)b_2}{c_2}\right]^{1/m} \eta_1^{s/m} \le \eta_2 \le \left[\frac{b_1}{(1+\delta)c_1}\right]^{1/l} \eta_1^{r/l}.$$

In view of $ls = rm$, it suffices to find a suitable δ so that

$$(1+\delta)^{l+m} \le \frac{b_1^m c_2^l}{c_1^m b_2^c}.$$

By the assumption of the theorem, we can always choose δ small such that the inequality above holds. This ends the proof of Theorem 2(1). ☐

To prove Theorem 2(2), we need to establish the following lemmas.

Lemma 17. *Under the assumptions of Theorem* 2(2)*, i.e.,* $ls = rm$ *and* $\min\{a_1, a_2\} > \lambda_1$*, the solution of* (1) *satisfies* $(u, v) \ge (k_1 \Phi, k_2 \Phi)$ *for suitable constants* $k_1, k_2 > 0$.

Proof. By the assumption (2), there exists a positive constant k such that

$$(u_0, v_0) \ge (k_1 \Phi(x), k_2 \Phi(x)) \quad \text{for } x \in \bar{\Omega}.$$

Let $(\underline{u}, \underline{v}) = (k_1 \Phi(x), k_2 \Phi(x))$ with positive constant $k_1, k_2 \le k$. A direct calculation yields

$$\begin{aligned} &\Delta \underline{u} + \underline{u}(a_1 - b_1 \underline{u}^r + c_1 \underline{v}^l) = k_1 \Phi(-\lambda_1 + a_1 - b_1 k_1^r \Phi^r + c_1 k_2^l \Phi^l), \\ &\Delta \underline{v} + \underline{v}(a_2 + b_2 \underline{u}^s - c_2 \underline{v}^m) = k_2 \Phi(-\lambda_1 + a_2 + b_2 k_1^s \Phi^s - c_2 k_2^m \Phi^m). \end{aligned}$$

Next we show that there exist positive constants $k_1, k_2 \le k$ such that

$$-b_1 k_1^r \Phi^r + c_1 k_2^l \Phi^l \ge 0, \qquad b_2 k_1^s \Phi^s - c_2 k_2^m \Phi^m \ge 0.$$

In view of $ls = rm$, it is equivalent to show that for $\Phi > 0$

$$\frac{b_1^s \Phi^{rs}}{c_1^s \Phi^{ls}} \le \frac{k_2^{ls}}{k_1^{rs}} \le \frac{b_2^r \Phi^{sr}}{c_2^r \Phi^{rm}}. \tag{19}$$

Since $b_1^s c_2^r \le b_2^r c_1^s$, it is clear that we can find suitable $k_1, k_2 \le k$ such that (19) holds. Note that $\min\{a_1, a_2\} > \lambda_1$; thus

$$\begin{cases} \underline{u}_t \le \underline{u}^p \Delta \underline{u} + \underline{u}^{p+1}(a_1 - b_1 \underline{u}^r + c_1 \underline{v}^l) & \text{in } \Omega \times (0, T), \\ \underline{v}_t \le \underline{v}^q \Delta \underline{v} + \underline{v}^{q+1}(a_2 + b_2 \underline{u}^s - c_2 \underline{v}^m) & \text{in } \Omega \times (0, T), \\ \underline{u}(x, t) = \underline{v}(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ \underline{u}(x, 0) = k_1 \Phi(x) \le u_0(x), \qquad \underline{v}(x, 0) = k_2 \Phi(x) \le v_0(x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

It follows from Comparison Principle 1 that $(u, v) \ge (k_1 \Phi(x), k_2 \Phi(x))$. This ends the proof. ☐

Lemma 18. *Assume that* $0 < \alpha < 1$ *and* $d, \delta > 0$ *and that* w *is a classical solution of*

$$\begin{cases} w_t = \mathrm{d}w^{\alpha}(\Delta w + aw) & \text{in } \Omega \times (0,T), \\ w(x,0) = \delta & \text{for } x \in \bar{\Omega}, \\ w(x,t) = \delta & \text{on } \partial\Omega \times (0,T). \end{cases}$$

If $a > \lambda_1$*, then* $w(x,t)$ *blows up in finite time.*

Proof. See Lemma 3.1 in [10]. ☐

Proof (*Proof of Theorem* 2(2)). In view of $\min\{a_1, a_2\} > \lambda_1$, we can choose a smooth subdomain $\Omega_* \subset \Omega$ such that $\lambda_1 < \lambda_1^* \le \min\{a_1, a_2\}$, where λ_1^* is the first eigenvalue of $-\Delta$ in Ω_* with homogeneous Dirichlet boundary condition.

Define $\delta = \min\{k_1 \min_{\bar{\Omega}_*} \Phi(x), k_2 \min_{\bar{\Omega}_*} \Phi(x)\}$; then $\delta > 0$ and we consider the following problem:

$$\begin{cases} \underline{u}_t = \underline{u}^p \Delta \underline{u} + \underline{u}^{p+1}(a_1 - b_1 \underline{u}^r + c_1 \underline{v}^l) & \text{in } \Omega_* \times (0,T^*), \\ \underline{v}_t = \underline{v}^q \Delta \underline{v} + \underline{v}^{q+1}(a_2 + b_2 \underline{u}^s - c_2 \underline{v}^m) & \text{in } \Omega_* \times (0,T^*), \\ \underline{u}(x,t) = \underline{v}(x,t) = \delta & \text{on } \partial\Omega_* \times (0,T^*), \\ \underline{u}(x,0) = \delta, \qquad \underline{v}(x,0) = \delta & \text{for } x \in \bar{\Omega}_*. \end{cases} \tag{20}$$

Similarly, we can show $(u,v) \ge (\underline{u}, \underline{v}) \ge (\delta, \delta)$ on $\bar{\Omega}_* \times [0,T)$. Let $(\underline{u}, \underline{v}) = (l_1 w, l_2 w)$ where l_1, l_2 are some positive constants to be chosen later and w is a nonnegative function which will be defined later. We show that we can choose l_1, l_2 and w so that $(\underline{u}, \underline{v})$ is a subsolution of (20); moreover, w blows up in finite time.

Clearly $(l_1 w, l_2 w)$ is a subsolution of (20) if

$$\begin{cases} w_t \le l_1^p w^p [\Delta w + w(a_1 - b_1 l_1^r w^r + c_1 l_2^l w^l)] & \text{in } \Omega_* \times (0,T^*), \\ w_t \le l_2^q w^q [\Delta w + w(a_2 + b_2 l_1^s w^s - c_2 l_2^m w^m)] & \text{in } \Omega_* \times (0,T^*), \\ l_1 w(x,t) \le \delta, \qquad l_2 w(x,t) \le \delta & \text{on } \partial\Omega_* \times (0,T^*), \\ l_1 w(x,0) \le \delta, \qquad l_2 w(x,0) \le \delta & \text{for } x \in \bar{\Omega}_*. \end{cases} \tag{21}$$

As before, we can choose l_1, l_2 such that

$$-b_1 l_1^r w^r + c_1 l_2^l w^l \ge 0, \qquad b_2 l_1^s w^s - c_2 l_2^m w^m \ge 0.$$

Thus if we let $a = \min\{a_1, a_2\}$ and $d = \min\{l_1^p \delta^{p-\alpha}, l_2^q \delta^{q-\alpha}\}$ for some constant α and w is the classical solution of

$$\begin{cases} w_t = \mathrm{d}w^{\alpha}(\Delta w + aw) & \text{in } \Omega_* \times (0,T), \\ w(x,0) = \delta & \text{for } x \in \bar{\Omega}_*, \\ w(x,t) = \delta & \text{on } \partial\Omega_* \times (0,T), \end{cases}$$

then $(l_1 w, l_2 w)$ is a subsolution of (20). In view of Lemma 18, $(\underline{u}, \underline{v})$ blows up in finite time in $\bar{\Omega}_*$; the conclusion of the theorem follows. ☐

Proof (*Proof of Theorem* 3). Let $ls = r_0 m$ for some constant $r_0 > r$. Any nonnegative solution of (1) is clearly a supersolution of the following system:

$$\begin{cases} u_{1t} = u_1^p \Delta u_1 + u_1^{p+1}(a_1 - b_1 - b_1 u_1^{r_0} + c_1 v_1^l) & \text{in } \Omega \times \mathbb{R}^+, \\ v_{1t} = v_1^q \Delta v_1 + v_1^{q+1}(a_2 + b_2 u_1^s - c_2 v_1^m) & \text{in } \Omega \times \mathbb{R}^+, \\ u_1(x,t) = v_1(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_1(x,0) = u_0(x), \qquad v_1(x,0) = v_0(x) & \text{for } x \in \Omega. \end{cases} \tag{22}$$

For system (22) we can apply the same argument as in the proof of Theorem 2(2) and we conclude that the solution blows up for $b_1^s c_2^r \le b_2^r c_1^s$ and $\min\{a_1 - b_1, a_2\} > \lambda_1$. Similarly, if $\min\{a_1, a_2 - b_2\} > \lambda_1$, we reach the same conclusion by modifying the second equation in the system. ☐

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