Remarks on large solutions of a class of semilinear elliptic equations

Peng Feng
Department of Physical Sciences and Mathematics,
Florida Gulf Coast University,
Fort Myers, FL, USA
(pfeng@fgcu.edu)

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In this paper, we show existence, uniqueness and exact asymptotic behavior of solutions near the boundary to a class of semilinear elliptic equations $-\Delta u = \lambda g(u) - b(x)f(u)$ in Ω, where λ is a real number, $b(x) > 0$ in Ω and vanishes on $\partial\Omega$. The special feature is to consider $g(u)$ and $f(u)$ to be regularly varying at infinity and $b(x)$ is vanishing on the boundary with a more general rate function. The vanishing rate of $b(x)$ determines the exact blow-up rate of the large solutions. And the exact blow-up rate allows us to obtain the uniqueness result.

1. Introduction

This paper is concerned with the study of semilinear elliptic problems with boundary blow-up of the form

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, and $b(x) \in C^\alpha(\bar{\Omega}, \mathbb{R}^+)$ for some $\alpha \in (0,1)$, $\mathbb{R}^+ := [0, +\infty)$. A solution of (1.1) is called large (or explosive) solution, by which we mean a function $u \in C^2(\Omega)$ such that

$$u(x) \to +\infty \text{ as } d(x) := dist(x, \partial\Omega) \to 0^+.$$

Our main objectives here are to study the existence, uniqueness and asymptotic behavior of large solutions. We consider the following assumptions on $b(x)$

B1 $b(x) = 0$ on $\partial\Omega$ and there exists a positive increasing function $h \in C^1(0, \delta_0)$ for some $\delta_0 > 0$ such that

$$\lim_{d(x)\to 0^+} \frac{b(x)}{h^2(d(x))} = c_0 > 0,$$

and

$$\lim_{d\to 0^+} \frac{\int_0^d h(s)ds}{h(d)} = 0, \ \lim_{d\to 0^+} \left(\frac{\int_0^d h(s)ds}{h(d)} \right)' = l_1 > 0.$$

Remark 1.1. *This assumption on $b(x)$ includes many different vanishing behaviors including $h(d) = d^\nu$ which was studied by several authors. This type of $b(x)$ can also be found in [9, 10].*

We consider the following assumptions on $f \in C^1[0, +\infty)$

F1 $f(0) = 0,\ f' \geq 0,\ f'(0) = 0.$

F2 $f(t)/t$ is increasing on $(0, +\infty)$.

F3 f is regularly varying at infinity with index $p > 1$.

and the following assumptions on $g(t) \in C^1[0, +\infty)$

G1 $g(t) \geq 0$ is increasing on $(0, +\infty)$ and $\lim_{t\to 0^+} g'(t) > 0$.

G2 $g(t)/t$ is nonincreasing on $(0, +\infty)$.

G3 $g(t)$ is regularly varying at infinity with index $0 < q < 1$.

Moreover, assumptions F(1-3) and G(1-3) imply

H1 $f(t)/g(t)$ is increasing for all $t > 0$ and $\lim_{t\to 0^+} f(t)/g(t) = 0$.

The study of large solutions goes back to 1916 by Bieberbach [2] on the equation $\Delta u = e^u$ on a smooth bounded domain in $\mathbb{R}^2$. Problem of this type arises in Riemannian geometry. The result was extended to smooth bounded domains in $\mathbb{R}^3$ by Rademacher [20]. Large solutions of more general elliptic equation $\Delta u = f(u)$ in n-dimensional domains were studied by Keller [12] and Osserman [18]. More precisely, they obtained the following necessary and sufficient condition for the existence of large solution

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \text{ where } F(t) = \int_0^t f(s)ds$$

provided that f satisfies

$$f \in C^1[0, \infty), f(0) = 0, f(s) > 0 \text{ for } s > 0 \text{ and } f'(s) \geq 0 \text{ for } s \geq 0.$$

The question of blow-up rates near $\partial\Omega$ and uniqueness of solutions appears in more recent literature. For example, Loewner and Nirenberg [16] studied the uniqueness and blow-up rate at the boundary for the elliptic equation $\Delta u = u^p$ where $p = \frac{N+2}{N-2}$ for $N > 2$. Bandle and Marcus [1] studied the uniqueness and asymptotic behavior near the boundary of a large solution for the more general equation $\Delta u = g(x, u)$ which includes the case $g(x, u) = b(x)u^p$ where $p > 1$ and $b(x)$ is positive continuous function in $\bar{\Omega}$ and b and $1/b$ are both bounded. Similar problem but for more general elliptic operators has also been studied in [21].

For $\Delta u = b(x)u^p$, most literature treated the case when $b(x)$ is bounded away from zero in $\bar{\Omega}$ in which large positive constants provide us with a priori bounds for the underlying Dirichlet boundary value problem. It has only been noticed recently by Lair [13, 14] that even when $b(x)$ vanishes on the boundary, large solution can still exist. One result they showed is that if $0 < p \leq 1$, then this equation has no large solution.

The problem of conformal deformation of metric with prescribed scalar curvature for a class of simple Riemannian manifold leads to the study of (1.1). Only until recently was the case of degenerate logistic type considered, which allows $b(x)$ to vanish on $\bar{\Omega}$, see for example [5, 6, 7, 8, 9, 10, 15, 17, 18] and references therein. However, many of them are restricted to the case $g(u) = u$(but see [8]), $b(x) = C_0 d^\nu(x) + o(d(x))$, and $f(u) = u^p$. Note that in this paper we extend the previous results in all three directions. We extend $g(u)$ to a more general class of functions which include $g(u) = u^q$, $0 < q < 1$, $b(x)$ assumes more general vanishing rate and $f(u)$ can be more general including the power function $f(u) = u^p$ for $p > 1$. We shall mention that in [5] the authors considered the special case when $g(u) = u$. In our case, it requires more subtle analysis when we derive the comparison principles and construct the sub- and supersolution. This is a continuous study of [9] in which the blow-up rate was obtained for $0 < q < 1$, $f(u) = u^p$, $b(x) = C_0 d^\nu(x) + o(d(x))$ and the domain is radial.

In this paper, applying Karamata regular variation theory, perturbed method and constructing sub- and supersolution, we show asymptotic behavior of solutions near the boundary. The exact blow-up rate ensures the uniqueness. Our main result is the following theorem.

Theorem 1.2. *Suppose f and g satisfy (F1-3), (G1-3) and $b(x)$ satisfies B1. Then for any $\lambda \in \mathbb{R}$, equation (1.1) admits a unique large solution u. Moreover, we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{Z(d(x))} = M,$$

where

$$M := \left[\frac{2 + l_1(p-1)}{c_0(p+1)}\right]^{1/(p-1)},$$

and the function $Z(t)$ is defined through

$$\int_{Z(t)}^{\infty} \frac{1}{\sqrt{2F(s)}} ds = \int_0^t h(s) ds, \;\; t \in (0, \delta_0).$$

Corollary 1.3. *Let $f(u) = u^p$, $p > 1$, $g(u) = u^q$, $0 < q < 1$, and $b(x) \sim c_0(d^\nu(x))$, then $h(d) = d^{\nu/2}(x)$, $l_1 = \frac{2}{\nu+2}$ and*

$$Z(t) = \left(\frac{p-1}{\nu+2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} t^{\frac{\nu+2}{1-p}}.$$

Any solution u to (1.1) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{d^{-\alpha}(x)} = M$$

where $\alpha = \frac{\nu+2}{p-1}$, $M = \left[\frac{\alpha(\alpha+1)}{c_0}\right]^{\frac{1}{p-1}}$.

Remark 1.4. *Our result in the corollary agrees with the result found in [7, 8, 9].*

Remark 1.5. *For the special case when $\lambda \neq 0$(in fact, it can be any bounded function) and $b(x)$ are bounded away from zero and $f(u) = u^p$, $g(u) = u^q$, we make the following observations:*
1. When $\lambda(x) > 0$, $0 < p < 1$, there is no large solution. This follows directly from Lair [13, 14].
2. When $\lambda(x) < 0$, $max\{p, q\} \leq 1$, there is no large solution. This follows from Lair as well.
3. If $p > q$ and $p > 1$, large solution exists for all $\lambda \neq 0$. This follows from the result of Bandle and Marcus [1].

The plan of this paper is as follows. In the next section we present some useful definitions and properties from regular variation theory. We als discuss some related properties associated with the main theorem. In the third section, we use the perturbed method and a general comparison principle to prove the existence of large solutions. The blow-up rate is studied in the fourth section. Finally we demonstrate some numerical computation to illuminate our result. We also remark on a simple way to find the blow-up rate to some similar equations.

2. Some Preliminary Study

In this section we give some preliminary considerations on various assumptions and properties needed for our main result. We start with some basic definitions and properties of regular variation theory which was initiated by Jovan Karamata in a well-known paper of 1930 [11]. For more information on this topic, we refer the readers to the book by Bingham et. al. [3].

Definition 2.1. *A positive measurable function f defined on $[a, \infty)$ for some $a > 0$, is called regularly varying at infinity with index $p \in \mathbb{R}$, written as $f \in \mathbb{R}_p$, if for all $\xi > 0$*

$$\lim_{t\to\infty} f(\xi t)/f(t) = \xi^p.$$

Definition 2.2. *A positive measurable function L defined on $[a, \infty)$ for some $a > 0$, is called slowly varying at infinity if for all $\xi > 0$*

$$\lim_{t\to\infty} L(\xi t)/L(t) = 1.$$

It follows by the definitions that any function $f \in \mathbb{R}_p$ can be represented in terms of a slowly varying function, $f(t) = t^p L(t)$.

Example 2.3. *The following examples are regularly varying at ∞ with index p*

$$t^p,\ t^p \ln(1+t),\ (t\ln(1+t))^p,\ t^p \ln(\ln(e+t)).$$

But $2 + \sin t$ clearly is not regularly varying.

Lemma 2.4. *(Representation theorem) The function L is slowly varying at infinity if and only if it can be written in the form*

$$L(t) = c(t)exp\left(\int_a^t \frac{y(s)}{s} ds\right),\ t \geq a,$$

for any $a > 0$, where $c(t)$ and $y(t)$ are measurable and as $t \to \infty$, $y(t) \to 0$ and $c(t) \to c > 0$.

We have the following useful properties on slowly varying function $L(t)$.

Lemma 2.5. *For any* $\alpha > 0$, $t^{\alpha}L(t) \to \infty$, *and* $t^{-\alpha}L(t) \to 0$ *as* $t \to \infty$

The following result of Karamata is often applicable. It essentially says that integrals of regularly varying functions are again regularly varying, or more precisely, one can take the slowly varying function out of the integral.

Lemma 2.6. *(Karamata's theorem) Let* $L(t)$ *be slowly varying and locally bounded in* $[a, \infty)$ *for some* $a \geq 0$. *Then*
a) for $p > -1$,

$$\int_a^t s^p L(s) ds \sim (p+1)^{-1} t^{p+1} L(t), \ t \to \infty,$$

b) for $p < -1$,

$$\int_t^{\infty} s^p L(s) ds \sim -(p+1)^{-1} t^{p+1} L(t), \ t \to \infty.$$

Remark 2.7. *The result remains true for* $p = -1$ *in the sense that*

$$\lim_{t\to\infty} \frac{1}{L(t)} \int_a^t \frac{L(s)}{s} ds = \infty.$$

Lemma 2.8. *Assume* f *satisfies (F1-2), then the following are equivalent*

$$(i) f \in \mathbb{R}_p; \ (ii) \lim_{t\to\infty} \frac{tf'(t)}{f(t)} = p; \ (iii) \lim_{t\to\infty} \left(\frac{F(t)}{f(t)}\right)' = (1+p)^{-1}. \tag{2.1}$$

Next we collect some properties on $Z(t)$ defined in Theorem 1.2. These properties can also be found in [5]. For the convenience of the reader, we include the proof.

Lemma 2.9. *If* $f(t)$ *satisfies (F1-3), then* $Z(t)$ *in Theorem 1.2 has the following properties:*

1. $\lim_{t\to 0^+} Z(t) = \infty$.
2. $\lim_{t\to 0^+} \frac{Z''(t)}{h^2(t) f(\xi Z)} = \frac{1}{\xi^p} \frac{pl_1 + 2 - l_1}{p+1}$ *for any* $\xi > 0$.
3. $\lim_{t\to 0^+} \frac{Z(t)}{Z''(t)} = \lim_{t\to 0^+} \frac{Z(t)}{Z'(t)} = \lim_{t\to 0^+} \frac{Z'(t)}{Z''(t)} = 0$.
4. $\lim_{t\to 0^+} \frac{g(\xi Z(t))}{Z''(t)} = 0$.

Proof. (1) This property follows directly from the definition of $Z(t)$.

(2) Here we only check for $\xi = 1$ since $f \in \mathbb{R}_p$ with $p > 1$. From the definition, we have the following

$$Z'(t) = -h(t)\sqrt{2F(Z)}$$

and

$$Z''(t) = h^2(t) f(Z(t)) \left(1 - 2\frac{h' \int_0^t h(s) ds}{h^2} \frac{\sqrt{F(Z)}}{f(Z) \int_Z^{\infty} [F(s)]^{-1/2} ds}\right)$$

for any $t \in (0, \delta_0)$. Applying l'Hôpital's rule and Lemma 2.8 we have

$$\lim_{t\to 0^+} \frac{\sqrt{F(Z)}}{f(Z)\int_Z^\infty [F(s)]^{-1/2}ds} = \frac{p-1}{2(p+1)}.$$

We also have

$$\lim_{t\to 0^+} \frac{h'(t)\int_0^t h(s)ds}{h^2(t)} = 1 - l_1.$$

Thus it follows that $\lim_{t\to 0^+} Z''(t)/h^2(t)f(\xi Z(t)) = \frac{1}{\xi^p}\frac{2+l_1(p-1)}{p+1}$.

(3) Note that

$$\lim_{t\to 0^+} \frac{Z'(t)}{h^2(t)f(Z(t))} = -\lim_{t\to 0^+} \frac{\sqrt{2F(Z(t))}}{h(t)f(Z(t))} = \lim_{t\to 0^+} \frac{\int_0^t h(s)ds}{h(t)} \frac{\sqrt{2F(Z(t))}}{f(Z(t))\int_Z^\infty [2F(s)]^{-1/2}ds} = 0$$

Hence $lim_{t\to 0^+} Z'(t)/Z''(t) = 0$. By a similar process and l'Hôpital's rule, we can also obtain $\lim_{t\to 0^+} Z(t)/Z'(t) = 0$ and $\lim_{t\to 0^+} Z(t)/Z''(t) = 0$.

(4) By the assumption on $g(t)$, we may represent it in terms of a slowly varying function $L(t)$, combining property (3) and Lemma 2.5, we have

$$\begin{aligned}\lim_{t\to 0^+} \frac{g(\xi Z(t))}{Z''(t)} &= \lim_{t\to 0^+} \frac{\xi^q Z^q(t) L(\xi Z(t))}{Z''(t)} = \lim_{t\to 0^+} \frac{\xi^q Z^q(t) L(Z(t))}{Z''(t)} \frac{L(\xi Z(t))}{L(Z(t))} \\ &= \lim_{t\to 0^+} \frac{\xi^q Z(t)}{Z''(t)} \frac{L(Z(t))}{Z^{1-q}(t)} = 0.\end{aligned} \tag{2.2}$$

□

We consider the function $\psi(t)$ defined by

$$\psi(t) := Af(t) - \beta g(t)$$

for certain constants $A > 0$ and $\beta > 0$ to be chosen later. Clearly $\psi(t) \in C^1((0, +\infty), [0, +\infty))$,

$$\lim_{t\to 0^+} \psi(t) = -\beta g(0) \le 0$$

and

$$\lim_{t\to +\infty} \psi(t) = +\infty.$$

Moreover, thanks to (F1) and (G1), we have $\lim_{t\to 0^+} \psi'(t) < 0$. Actually,

$$\lim_{t\to 0^+} \psi(t)/g(t) = \lim_{t\to 0^+} (Af(t)/g(t) - \beta) = -\beta.$$

Hence there exists a unique t_0 such that $\psi(t_0) = 0$ and $\psi(t) > 0$ for all $t > t_0$. Moreover, due to (H1), we have $f'(t)/g'(t) > f(t)/g(t)$ for all $t > 0$, thus for $t > t_0$

$$\psi'(t) = Af'(t) - \beta g'(t) > g'(t)(Af(t)/g(t) - \beta) = \frac{g'(t)}{g(t)}\psi(t) > 0$$

as $\psi(t) > 0$ for all $t > t_0$.

We now prove the following result.

Lemma 2.10. *Suppose f satisfies (F1-3) and g satisfies (G1-3). Then for each $t > t_0$,*

$$I(t) := \int_t^\infty \left[\int_t^\tau \psi(s) ds \right]^{-1/2} d\tau < \infty,$$

and

$$\lim_{t \to t_0^-} I(t) = \infty.$$

Proof. First we note that condition F3 implies

$$\int_a^\infty \left(\int_0^\tau f(s) ds \right)^{-1/2} d\tau < \infty. \tag{2.3}$$

We consider

$$G(\tau) := \int_t^\tau \psi(s) ds, \quad \tau \geq t > t_0.$$

Then $G(t) = 0$ and $G'(\tau) = \psi(\tau) > 0$ since $\tau \geq t > t_0$ and $\psi(\tau) > 0$ if $\tau > t_0$. Hence,

$$\lim_{\tau \to t} \frac{G(\tau)}{\tau - t} = \lim_{\tau \to t} \frac{G(t) + G'(t)(\tau - t) + o(\tau - t)}{\tau - t} = \psi(\tau) > 0. \tag{2.4}$$

Moreover, we have

$$\lim_{\tau \to \infty} \frac{G(\tau)}{\int_0^\tau f(s) ds} = \lim_{\tau \to \infty} \frac{\psi(\tau)}{f(\tau)} = A. \tag{2.5}$$

Combining (2.3), (2.4), (2.5), and by the comparison test for improper integrals, it follows that $I(t) < \infty$ for all $t > t_0$.

$\lim_{t \to t_0^-} I(t) = \infty$ can be obtained by the fact that

$$\lim_{\tau \to t_0} \int_{t_0}^\tau \psi(s) ds = 0, \text{ and } \frac{d}{d\tau} \int_{t_0}^\tau \psi(s) ds|_{\tau = t_0} = \psi(t_0) = 0.$$

□

3. Existence Result

The following comparison principle is essential in obtaining the existence result. Our proof involves a simple "energy" device that can be found in [1, 6, 7].

Lemma 3.1. *Let Ω_0 be a smooth bounded domain in $\mathbb{R}^N$. Assume $f(u)$ satisfies F(1-2) and $g(u)$ satisfies G(1-2), $b(x)$, $r(x)$ are C^α functions on $\bar{\Omega}_0$ such that $r(x) \geq 0$, $b(x) > 0$ on Ω_0 and $\lambda \in R$. Let $u_1,\ u_2 \in C^2(\Omega_0)$ be positive functions such that*

$$-\Delta u_1 - \lambda g(u_1) + b(x) f(u_1) - r(x) \geq 0 \geq -\Delta u_2 - \lambda g(u_2) + b(x) f(u_2) - r(x) \text{ in } \Omega_0 \tag{3.1}$$

and

$$limsup_{d(x) \to 0} (u_1(x) - u_2(x)) \geq 0,$$

where $d(x) := dist(x, \partial\Omega_0)$. Then $u_1 \geq u_2$ in Ω_0.

Proof. First we consider the case when $\lambda \geq 0$. It follows from (3.1) that for any nonnegative function $\Phi \in H^1(\Omega_0)$ with compact support, we have

$$\int_{\Omega_0} \nabla u_1 \nabla \Phi - \lambda g(u_1)\Phi + b(x) f(u_1)\Phi - r\Phi \geq 0 \geq \int_{\Omega_0} \nabla u_2 \nabla \Phi - \lambda g(u_2)\Phi + b(x) f(u_2)\Phi - r\Phi \tag{3.2}$$

Let $\epsilon_1 > \epsilon_2 > 0$ and denote

$$\Omega_+(\epsilon_1, \epsilon_2) = \{x \in \bar{\Omega}_0 : u_2(x) + \epsilon_2 > u_1(x) + \epsilon_1\},$$

and

$$v_i = (u_i + \epsilon_i)^{-1}[(u_2(x) + \epsilon_2)^2 - (u_1(x) + \epsilon_1)^2]^+.$$

Notice that $v_i \in H^1(\Omega_0)$ and it has compact support in Ω_0 (it vanishes outside Ω_+). Replacing Φ in (3.2) by v_1, v_2 and applying integration by parts and subtraction yields

$$\int_{\Omega_+} (\nabla u_1 \nabla v_1 - \nabla u_2 \nabla v_2) - \int_{\Omega_+} \lambda(g(u_1)v_1 - g(u_2)v_2) \geq \int_{\Omega_+} b(x)(f(u_2)v_2 - f(u_1)v_1) + \int_{\Omega_+} r(x)(v_1 - v_2) \tag{3.3}$$

A simple calculation shows that the first integral on the left-hand side of (3.3) equals

$$-\int_{\Omega_+} \left(\left| \nabla u_2 - \frac{u_2 + \epsilon_2}{u_1 + \epsilon_1} \nabla u_1 \right|^2 + \left| \nabla u_1 - \frac{u_1 + \epsilon_1}{u_2 + \epsilon_2} \nabla u_2 \right|^2 \right) dx \leq 0.$$

As $0 < \epsilon_2 < \epsilon_1 \to 0$, the second term on the left-hand side of (3.3) equals

$$-\int_{\Omega_+(0,0)} \lambda[g(u_1)/u_1 - g(u_2)/u_2] dx \leq 0.$$

On the other hand, the first term on the right-hand side converges to

$$\int_{\Omega_+(0,0)} b(x) \left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right) (u_2^2 - u_1^2) dx > 0$$

while the second term converges to

$$\int_{\Omega_+(0,0)} r(x)(1/u_1 - 1/u_2)(u_2^2 - u_1^2) dx \geq 0.$$

Therefore, we should have a contradiction unless $\Omega_+(0,0)$ has measure 0, i.e.,$u_1 \geq u_2$ on Ω_0.

For the additional case when $\lambda < 0$, we suppose that there exists $x_0 \in \Omega_0$ such that

$$0 > u_1(x_0) - u_2(x_0) := min_{\bar{\Omega}_0}(u_1 - u_2).$$

Thus since $\lambda < 0$,

$$0 \geq -\Delta(u_1 - u_2)|_{x=x_0} \geq \lambda(g(u_1(x_0)) - g(u_2(x_0))) + b(x)(f(u_2(x_0)) - f(u_1(x_0))) > 0,$$

which is impossible. $\square$

Our next lemma shows the uniform boundness of an auxiliary problem. A similar lemma where $g(u) = u^q$ can be found in [8, 17].

Lemma 3.2. *Let $B(R) \subset \mathbb{R}^n$ be an arbitrary ball centered at x_0 and consider the auxiliary problem*

$$\begin{cases} -\Delta u = \lambda g(u) - Af(u) & \text{in } B, \\ u = \tau & \text{on } \partial B, \end{cases} \tag{3.4}$$

where $\lambda \in \mathbb{R}$, $A > 0$ and $\tau > t_0$. Then there exists a constant $M := M(R)$ such that any solution u_τ of (3.4) satisfies $||u_\tau||_{C(B)} \leq M$.

Proof. For each $x \in B$, we denote

$$u(x) := \Psi_\tau(r), \ r := |x - x_0|,$$

where Ψ_τ solves

$$\begin{cases} \Psi_\tau''(r) + \frac{N-1}{r}\Psi_\tau'(r) = h(\Psi_\tau), & r \in (0, R) \\ \Psi_\tau'(0) = 0, \ \Psi_\tau(R) = \tau. \end{cases} \tag{3.5}$$

Since $\tau > t_0$, it is easy to see that $\Psi_\tau > t_0$, $h(\Psi_\tau) > 0$, $h''(\Psi_\tau) > 0$. The function Ψ_τ satisfies

$$\left(r^{N-1}\Psi_\tau'(r)\right)' = r^{N-1}h(\Psi_\tau(r)). \tag{3.6}$$

Integrating (3.6) from 0 to r yields

$$\Psi_\tau'(r) = r^{1-N} \int_0^r s^{N-1}h(\Psi_\tau(s))ds > 0. \tag{3.7}$$

Thus

$$\frac{d}{dr}h(\Psi_\tau(r)) = h'(\Psi_\tau)\Psi_\tau' > 0.$$

It follows from (3.7) that

$$\Psi_\tau'(r) \leq r^{1-N}h(\Psi_\tau(r)) \int_0^r s^{N-1}ds = \frac{r}{N}h(\Psi_\tau(r)).$$

Hence

$$\Psi_\tau'' = h(\Psi_\tau) - \frac{N-1}{r}\Psi_\tau' \geq \frac{1}{N}h(\Psi_\tau).$$

Moreover, since $\Psi_\tau' \geq 0$, we have

$$\frac{1}{N}h(\Psi_\tau) \leq \Psi_\tau''(r) \leq h(\Psi_\tau). \tag{3.8}$$

Multiplying (3.8) by Ψ_τ' and integrating from 0 to r yields

$$\frac{2}{N}\int_{\Psi_\tau(0)}^{\Psi_\tau(r)} h(s)ds \leq [\Psi_\tau'(r)]^2 \leq 2\int_{\Psi_\tau(0)}^{\Psi_\tau(r)} h(s)ds. \tag{3.9}$$

Integrating the square root of the reciprocal of (3.9) gives

$$\frac{2}{N}\int_{\Psi_\tau(0)}^{\Psi_\tau(r)}\left[\int_{\Psi_\tau(0)}^{z} h(s)ds\right]^{-1/2} dz \le r \le \sqrt{N/2}\int_{\Psi_\tau(0)}^{\Psi_\tau(r)}\left[\int_{\Psi_\tau(0)}^{z} h(s)ds\right]^{-1/2} dz. \tag{3.10}$$

Thus

$$R \le \sqrt{N/2}\int_{\Psi_\tau(0)}^{\tau}\left[\int_{\Psi_\tau(0)}^{z} h(s)ds\right]^{-1/2} dz.$$

Applying Lemma 2.10, we obtain that $\Psi_\tau(0)$ must be bounded above by a constant M independent of τ. □

Now we are in position to prove the existence part of Theorem 1.2. Consider the following perturbed problem

$$\begin{cases} -\Delta u = \lambda g(u) - (b(x) + \frac{1}{n^\gamma}) f(u), & x \in \Omega, \\ u = n, & x \in \partial\Omega, \end{cases} \tag{3.11}$$

where $\gamma > 0$ satisfies $p > q + \gamma$. Since 0 is a subsolution and n is a supersolution for n sufficiently large, (3.1) admits a solution $u_n \in C^{2,\alpha}(\bar{\Omega})$ with $u_n \le n$. Moreover, Lemma 3.1 shows that $\{u_n\}_n$ is increasing. Our purpose is to pass to the limit as $n \to \infty$. Thanks to Lemma 3.2, u_n is uniformly bounded on every compact subdomain of Ω. By the monotonicity of $\{u_n\}$, we conclude $u_n \to \underline{u}$ in $L^\infty_{loc}(\Omega)$. Finally, standard elliptic regularity arguments lead to $u_n \to \underline{u}$ in $C^{2,\alpha}_{loc}(\Omega)$.

4. Blow-up Rate and Uniqueness

In this section, we establish the exact blow-up rate and obtain the uniqueness. We start with the following comparison lemma. The proof of the lemma is carried out by applying sub- and supersolution method in domain $\{x \in \Omega : d(x) > 1/n\}$ and passing $n \to +\infty$ through a diagonal process.

Lemma 4.1. *Suppose $\underline{u}$ and $\bar{u}$ satisfy*

$$-\Delta \underline{u} \le \lambda g(\underline{u}) - b(x) f(\underline{u}) \text{ in } \Omega,$$

$$-\Delta \bar{u} \ge \lambda g(\bar{u}) - b(x) f(\bar{u}) \text{ in } \Omega,$$

$\lim_{d(x)\to 0^+} \underline{u}(x) = \lim_{d(x)\to 0^+} \bar{u}(x) = \infty$ *and* $\underline{u} \le \bar{u}$ *in* Ω. *Then (1.1) admits a solution* $u \in C^2(\Omega)$ *satisfying* $\underline{u} \le u \le \bar{u}$ *in* Ω.

To prove the blow-up rate at the boundary, we construct the sub- and supersolutions with the same blow-up rate. To that aim, we define $\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$ and $\partial\Omega_\delta := \{x \in \Omega : d(x) = \delta\}$. By the regularity of $\partial\Omega$, we can choose δ sufficiently small so that

1. $d(x) \in C^2(\bar{\Omega}_{2\delta})$;
2. $h^2(d)$ is increasing on $(0, 2\delta)$;

3. $Z''(d) > 0$ for any $d \in (0, 2\delta)$;
4. $(c_0 - \epsilon)h^2(d(x)) < b(x) < (c_0 + \epsilon)h^2(d(x))$, for any $x \in \Omega_{2\delta}$. Here $0 < \epsilon < c_0/2$ is a fixed constant.

Define

$$\xi_2 = \left[\frac{2 + l_1(p-1)}{(c_0 - 2\epsilon)(p+1)}\right]^{\frac{1}{p-1}}, \quad \xi_1 = \left[\frac{2 + l_1(p-1)}{(c_0 + 2\epsilon)(p+1)}\right]^{\frac{1}{p-1}}.$$

Let $\mu \in (0, \delta)$ be arbitrary. We define

$$\bar{u}_\mu = \xi_2 Z(d(x) - \mu),\ x \in \Omega_{2\delta} \setminus \Omega_\mu,$$

and

$$\underline{u}_\mu = \xi_1 Z(d(x) + \mu),\ x \in \Omega_{2\delta - \mu}.$$

It follows from $|\nabla d(x)| = 1$ that

$$\begin{aligned}
&-\Delta \bar{u}_\mu - \lambda g(\bar{u}_\mu) + b(x) f(\bar{u}_\mu) \\
&= -\xi_2 Z''(d(x) - \mu) - \xi_2 Z'(d(x) - \mu) - \lambda g(\xi_2 Z(d(x) - \mu)) + b(x) f(\xi_2 Z(d(x) - \mu)) \\
&= -\xi_2 Z''(d(x) - \mu)\left[1 + \frac{Z'(d(x) - \mu)}{Z''(d(x) - \mu)} + \frac{\lambda g(\xi_2 Z(d(x) - \mu))}{\xi_2 Z''(d(x) - \mu))} - b(x)\frac{f(\xi_2 Z(d(x) - \mu)}{\xi_2 Z''(d(x) - \mu)}\right] \\
&\geq -\xi_2 Z''(d(x) - \mu)\left[1 + \frac{Z'(d(x) - \mu)}{Z''(d(x) - \mu)} + \frac{\lambda g(\xi_2 Z(d(x) - \mu))}{\xi_2 Z''(d(x) - \mu))} - (c_0 - \epsilon)\frac{h^2(d(x) - \mu) f(\xi_2 Z(d(x) - \mu)}{\xi_2 Z''(d(x) - \mu)}\right].
\end{aligned} \tag{4.1}$$

Applying Lemma 2.9 and by setting δ to be small, we see that

$$-\Delta \bar{u}_\mu - \lambda g(\bar{u}_\mu) + b(x) f(\bar{u}_\mu) \geq 0.$$

Similarly, we have

$$\begin{aligned}
&-\Delta \underline{u}_\mu - \lambda g(\underline{u}_\mu) + b(x) f(\underline{u}_\mu) \\
&= -\xi_1 Z''(d(x) + \mu) - \xi_1 Z'(d(x) + \mu) - \lambda g(\xi_1 Z(d(x) + \mu)) + b(x) f(\xi_1 Z(d(x) + \mu)) \\
&= -\xi_1 Z''(d(x) + \mu)\left[1 + \frac{Z'(d(x) + \mu)}{Z''(d(x) + \mu)} + \frac{\lambda g(\xi_1 Z(d(x) + \mu))}{\xi_1 Z''(d(x) + \mu))} - b(x)\frac{f(\xi_1 Z(d(x) + \mu)}{\xi_1 Z''(d(x) + \mu)}\right] \\
&\leq -\xi_1 Z''(d(x) + \mu)\left[1 + \frac{Z'(d(x) + \mu)}{Z''(d(x) + \mu)} + \frac{\lambda g(\xi_1 Z(d(x) + \mu))}{\xi_1 Z''(d(x) + \mu))} - (c_0 + \epsilon)\frac{h^2(d(x) + \mu) f(\xi_1 Z(d(x) + \mu)}{\xi_1 Z''(d(x) + \mu)}\right],
\end{aligned} \tag{4.2}$$

and

$$-\Delta \bar{u} - \lambda g(\bar{u}) + b(x) f(\bar{u}) \leq 0.$$

Let w be an arbitrary solution of

$$\begin{cases}
-\Delta u = \lambda g(u) - b(x) f(u), & x \in \Omega \setminus \bar{\Omega}_\delta, \\
u = 1, & x \in \partial\Omega, \\
u = +\infty, & x \in \partial\Omega_\delta.
\end{cases}$$

We see that

$$u + w|_{\partial\Omega} = \infty > \underline{u}_\mu|_{\partial\Omega},\ u + w|_{\partial\Omega_\delta} = \infty > \underline{u}_\mu|_{\partial\Omega_\delta},$$

$$\bar{u}_\mu + w|_{\partial\Omega_\mu} = \infty > u|_{\partial\Omega_\mu},\ \bar{u}_\mu + w|_{\partial\Omega_\delta} = \infty > u|_{\partial\Omega_\delta}.$$

Lemma 3.1 ensures that

$$\underline{u}_\mu \le u(x) + w(x),\ x \in \Omega_\delta;\ u(x) \le \bar{u}_\mu(x) + w(x),\ x \in \Omega_\delta \setminus \Omega_\mu.$$

Passing the limit $\mu \to 0^+$, we see that

$$\xi_1 Z(d(x)) \le u(x) + w(x) \le \xi_2 Z(d(x)) + 2w(x),\ x \in \Omega_\delta,$$

which implies

$$\xi_1 \le \liminf_{d(x)\to 0^+} \frac{u(x)}{Z(d(x))} \le \limsup_{d(x)\to 0^+} \frac{u(x)}{Z(d(x))} \le \xi_2.$$

Finally, we set $\epsilon \to 0$ to obtain the exact blow-up rate.

Proof. (*Uniqueness*) Let $u_1,\ u_2 \in C^2(\Omega)$ be two arbitrary large solutions, then the exact blow-up rate yields $\lim_{d(x)\to 0} u_1(x)/u_2(x) = 1$. Hence, for any $\epsilon \in (0,1)$, there exists $\delta > 0$ which depends on ϵ such that

$$(1-\epsilon)u_2 \le u_1 \le (1+\epsilon)u_2,\ x \in \Omega_\delta.$$

Clearly, u_1 is a positive solution of

$$\begin{cases} -\Delta u = \lambda g(u) - b(x) f(u), & x \in \Omega_\delta, \\ u = u_1, & x \in \partial(\Omega \setminus \Omega_\delta). \end{cases} \tag{4.3}$$

By the assumptions on f, g, we see that $u^- = (1-\epsilon)u_2$ and $u^+ = (1+\epsilon)u_2$ are positive subsolution and supersolution of (4.3). Thus (4.3) has a positive solution $\tilde{u}$ such that

$$(1-\epsilon)u_2 \le \tilde{u} \le (1+\epsilon)u_2.$$

Moreover, by comparison principle (Lemma 3.1), (4.3) admits a unique solution, i.e., $u_1 \equiv \tilde{u}$ in $\Omega \setminus \Omega_\delta$. Thus for $x \in \Omega \setminus \Omega_\delta$, we have

$$(1-\epsilon)u_2 \le u_1 \le (1+\epsilon)u_2.$$

Letting $\epsilon \to 0$, we see that $u_1 \equiv u_2$ in Ω. This concludes the proof of the uniqueness. □

5. Illustrative Computations and Final Remark

In this section, we consider the large solution to the following equation in a radial domain with radius $R = 1$ in $\mathbb{R}^N$

$$\Delta u = \frac{1}{6}(1-|x|)^2 u^3.$$

Clearly the radial solution $u(r) := u(|x|)$ satisfies

$$u''(r) + \frac{N-1}{r}u' = \frac{1}{6}(1-r)^2u^3, \text{in } (0,1)$$

with $u'(0) = 0$; large solutions are those with $u(r) \to \infty$ as $r \to 1^-$. Find radially symmetric large solutions is equivalent to finding initial condition $u(0) = p$ such that the solution to the following Cauchy problem

$$\begin{cases} u' = w, & u(0) = p, \\ w' = -\frac{N-1}{r}w + \frac{1}{6}(1-r)^2u^3, & w(0) = 0, \end{cases} \tag{5.1}$$

exists on the interval $[0, 1)$ and blows up at 1.

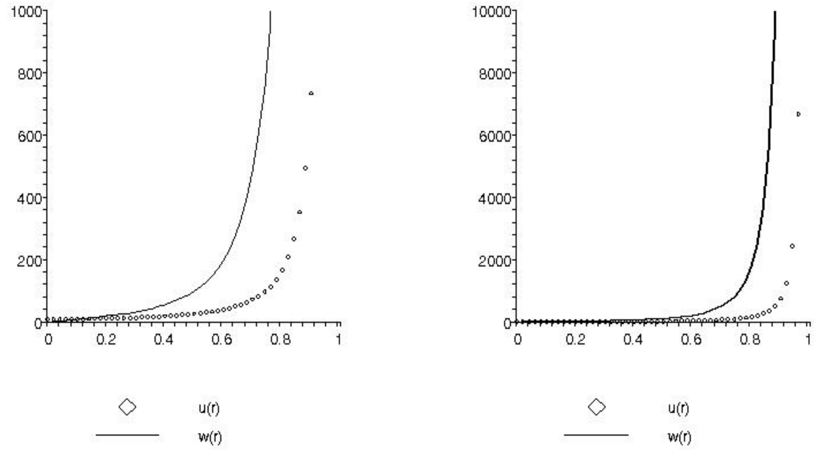

Figure 1. Large radial solution (u, w) with $u(0) > 0$ and $w(0) = 0$ of Problem (5.1) on a radial domain with $R = 1$, for $N = 2$(left) and $N = 6$(right)

Figure 1 shows the computed profiles of the nonnegative large solutions (u, w) of the problem (5.1) for two values of the space dimension $N = 2$ and $N = 6$. The way that these profiles have been calculated is through the following process.

Figure 2. Critical solutions $(A(r), B(r))$ of Problem (5.2) for $N = 2$, $p \approx 10.604330142$(left) and for $N = 6$, $p \approx 16.351198742$(right)

By Corollary 1.3, we obtain the following asymptotic behavior at $r = 1$.

$$u(r) \sim \frac{6}{(1-r)^2},\ w(r) \sim \frac{12}{(1-r)^3}.$$

We define functions $A(r)$ and $B(r)$ by

$$A(r) := \frac{(1-r)^2}{6}u(r),\ B(r) = \frac{(1-r)^3}{12}w(r);$$

then $(A(r), B(r))$ is a solution of

$$\begin{cases}(1-r)A'(r) = 2(B(r) - A(r)), & A(0) = \frac{1}{6}p, \\ (1-r)B'(r) = -\frac{N-1}{r}(1-r)B(r) - 3B(r) + 3A^3(r), & B(0) = 0.\end{cases} \tag{5.2}$$

System (5.2) is both singular at $r = 0$ and $r = 1$, but we still have well-posedness. Let the maximal interval of existence of this system to be $[0, R_p)$ and let $(A(r), B(r))$ be the corresponding solution of (5.2). Then if $R_p > 1$, then $(A(r), B(r)) \to (0, 0)$ as $r \to 1^-$. If $R_p < 1$, then $(A(r), B(r))$ ceases to exist before the singularity $r = 1$. If $R_p = 1$, numerical result shows that $(A(r), B(r)) \to (1, 1)$ as $r \to 1^-$. In fact, there exists a unique p such that $R_p = 1$ and the corresponding solution (u, w) blows up at 1.

One might ask what happens to the blow-up rate if q is any positive number rather than $0 < q < 1$. A natural way to see this is to consider the following one-dimensional problem with $g(u) = u^q$ and with a general $\lambda(x) \in L^\infty$.

$$\begin{cases}-u'' = \lambda(x)u^q - b(x)u^p & \text{in } (0,1) \\ \lim_{x\to 1^-} u(x) = +\infty \\ u(0) = 0\end{cases} \tag{5.3}$$

where $b(x) = C_0 d^\nu + o(d^\nu)$ as $d \to 0+$ with $\nu > 0$ and $C_0 > 0$.

By the assumption on b, we may write

$$b(x) = \beta(x)(1-x)^\nu, \ x \in (0,1), \ \beta(1) > 0.$$

To find out the blow-up rate, we substitute

$$u(x) = \phi(x)(1-x)^{-\alpha}, \ x \in (0,1), \ \alpha \geq 0, \ \phi(1) > 0$$

into (5.3), we have

$$\phi''(1-x)^{-\alpha} + 2\phi'\alpha(1-x) + \alpha(\alpha+1)\phi(1-x)^{-\alpha-2} = \beta(x)\phi^p(1-x)^{\nu-\alpha p} - \lambda(x)\phi^q(1-x)^{-\alpha q}. \tag{5.4}$$

Multiplying on both sides by $(1-x)^{\alpha+2}$ yields

$$\phi''(1-x)^2 + 2\phi'\alpha(1-x) + \alpha(\alpha+1)\phi = \beta(x)\phi^p(1-x)^{\nu-\alpha p+\alpha+2} - \lambda(x)\phi^q(1-x)^{-\alpha q+\alpha+2}. \tag{5.5}$$

Assuming that $\lim_{x\to 1^-}(1-x)^2\phi'' = \lim_{x\to 1^-}(1-x)\phi' = 0$ and passing the limit $x \to 1^-$, we impose the following conditions.

Case A: $-\alpha q + \alpha + 2 = 0$, $\nu - \alpha p + \alpha + 2 > 0$ and $\alpha(\alpha+1)\phi(1) = -\lambda(1)\phi^q(1)$. In this case, we conclude that only when $\lambda(1) < 0$, $q > \frac{2p+\nu}{\nu+2}$, the solution blows up at the boundary and the blow-up rate is

$$\alpha = \frac{2}{q-1} \text{ and } \phi(1) = \left[\frac{\alpha(\alpha+1)}{-\lambda(1)}\right]^{\frac{1}{q-1}}.$$

Case B:$-\alpha q + \alpha + 2 = 0$, $\nu - \alpha p + \alpha + 2 = 0$ and $\alpha(\alpha+1)\phi(1) = \beta(1) - \phi^p(1) - \lambda(1)\phi^q(1)$. In this case, we conclude that only when $q = \frac{2p+\nu}{\nu+2}$, the solution blows up at the boundary and the blow-up rate is

$$\alpha = \frac{2}{q-1} \text{ and } \phi(1) \text{ is determined by the third equation .}$$

Case C: $-\alpha q + \alpha + 2 > 0$, $\nu - \alpha p + \alpha + 2 = 0$ and $\alpha(\alpha+1)\phi(1) = \beta(1)\phi^p(1)$. In this case, we conclude that when $q < \frac{2p+\nu}{\nu+2}$, the solution blows up at the boundary and the blow-up rate is

$$\alpha = \frac{\nu+2}{p-1} \text{ and } \phi(1) = \left[\frac{\alpha(\alpha+1)}{\beta(1)}\right]^{\frac{1}{p-1}}.$$

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