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On the blow-up rate of large solutions for a porous media logistic equation on radial domain

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Abstract

In this paper we establish the exact blow-up rate of the large solutions of a porous media logistic equation. We consider the carrying capacity function with a general decay rate at the boundary instead of the usual cases when it can be approximated by a distant function. Obtaining the accurate blow-up rate allows us to establish the uniqueness result. Our result covers all previous results on the ball domain and can be further adapted in a more general domain.

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1. Introduction

In this work, we consider the singular boundary value problem

$$\begin{cases} -\Delta w^m = \lambda w - a(x)w^p & \text{in } \Omega, \\ w = \infty & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded domain of \mathbb{R}^N , $N \geq 1$, with boundary $\partial\Omega$ of class C^2 , $\lambda \in \mathbb{R}$, $m > 1$, $p > 1$ and $a(x) \in C^\alpha(\bar{\Omega}, \mathbb{R}^+)$ for some $\alpha \in (0, 1)$, $\mathbb{R}^+ := [0, +\infty)$. The solutions are often

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known as large solutions. More precisely, by a large solution we mean any classical solution w such that

$$w(x) \rightarrow +\infty \quad \text{as } d(x) := \text{dist}(x, \partial\Omega) \rightarrow 0^+.$$

The special case $m = 1$ is related to some prescribed curvature problem in Riemannian geometry. If $\lambda = 0$, $a(x) = 1$, it is easy to check that $f(t) = t^p$ with $p > 1$ satisfies the well-known Keller–Osseman condition

$$\int_{t_0}^{\infty} \left\{ \int_0^t f(s) ds \right\}^{-1/2} dt < +\infty \quad \text{for all } t_0 > 0,$$

which is a necessary condition for the existence of large solutions. Indeed, the existence of a large solution can be established by the method of supersolution and subsolution provided that the domain Ω is regular enough, see, e.g., [8]. The uniqueness was established by Marcus and Veron [10,11] in very general domain for all $p > 1$. However, the existence was not established for such general domain whose boundary is locally represented as a graph of a continuous function. The existence for general bounded domain was recently obtained by Kim [9] under the restriction that $p \in (1, \frac{n}{n-2})$ for $n \geq 3$ and $p \in (1, \infty)$ for $n = 2$. The same author also established the uniqueness under the assumption that $\partial\Omega = \partial\bar{\Omega}$.

For $m = 1$, $\lambda = 0$, $a(x) \geq 0$ and $a(x) \neq 0$, the existence was established in [6] under the assumption that there exist constant $C_1, C_2 > 0$ and $v_2 \geq v_1 > -2$ such that

$$C_2 d(x)^{v_2} \leq a(x) \leq C_1 d(x)^{v_1}, \quad x \in \Omega.$$

Uniqueness can be established for the case $v_1 = v_2$.

For $m = 1$, $\lambda = 0$, $a(x) \geq a_0 > 0$ in $\bar{\Omega}$, different types of equations were studied, e.g., [2]. In the case $a(x) \sim C_0 d(x)^v + o(d^v)$ as $x \rightarrow \partial\Omega$, blow-up rate and uniqueness were studied in [5].

For the porous media logistic equation, i.e., $m > 1$, $p > 1$, $\lambda \in \mathbb{R}$, the existence of large solutions was studied in [3,4]. Their result showed that large solution exist if, and only if, the nonlinear diffusion is not too large. Precisely, they proved the following theorem.

Theorem 1. [2, Theorem 5.2] *Suppose $\lambda \in \mathbb{R}$ and $a(x) \in C^\alpha(\bar{\Omega}, \mathbb{R}^+)$ satisfies the following assumption: The open set $\Omega_+ := \{x \in \partial\Omega; a(x) > 0\}$ is connected with boundary $\partial\Omega_+$ of class C^2 , and the open set $\Omega_0 := \Omega \setminus \bar{\Omega}_+$ satisfies $\bar{\Omega}_0 \subset \Omega$. Then:*

- (a) When $p > m > 1$, (1) possesses a positive solution.
- (b) When $m \geq p > 1$, (1) does not admit a positive solution.

However, in [5], to prove the uniqueness, it was assumed that

$a(x) = \beta d(x)^v [1 + \beta d(x) + o(d(x))]$ as $d(x) \rightarrow 0^+$ for some constants $\beta > 0$, $v \geq 0$ and hence

$$\lim_{x \rightarrow x_0} \frac{a(x)}{\beta d(x)^v} = 1 \quad \text{uniformly in } x_0 \in \partial\Omega.$$

And in [4], to prove the uniqueness, it was assumed that

$$\lim_{x \rightarrow x_0} \frac{a(x)}{\beta(x_0)d(x)^{\nu(x_0)}} = 1 \quad \text{uniformly in } x_0 \in \partial\Omega,$$

for some $\beta(x) \in C(\partial\Omega; \mathbb{R}_+)$ and $\nu(x) \in C(\partial\Omega, \mathbb{R}_+)$.

The purpose of this work is to consider more general $a(x)$ and Ω a ball B_R . Indeed, we shall assume $a(x) \in C(\Omega)$ and $a(x) = a(\|x\|) > 0$ in B_R . Furthermore, we assume $a(r)$ satisfies

$$\frac{\int_r^R a(s) ds}{a(r)} \in C^1([0, R]), \quad \lim_{r \rightarrow R} \frac{\int_r^R a(s) ds}{a(r)} = 0.$$

We shall study the exact blow-up rate at the boundary which helps us to establish the uniqueness result. By a similar process as in [4], we can extend the result to a more general domain Ω .

To analyze (1), we shall make the change of variable

$$u := w^m.$$

(1) becomes

$$\begin{cases} -\Delta u = \lambda u^{1/m} - a(x)u^{p/m} & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Our main result reads as follows.

Theorem 2. Consider the radially symmetric semilinear elliptic equation

$$\begin{cases} -\Delta u = \lambda u^{1/m} - a(x)u^{p/m} & \text{in } B_R(0), \\ u = \infty & \text{on } \partial B_R(0), \end{cases} \tag{3}$$

$p > m \geq 1$, $\lambda \in \mathbb{R}$, $a \in C([0, R]; [0, \infty))$ satisfying

$$a > 0 \quad \text{in } [0, R), \quad \frac{A(r)}{a(r)} \in C^1([0, R]),$$

$$\lim_{r \rightarrow R} \frac{A(r)}{a(r)} = 0 \quad \text{where } A(r) := \int_r^R a(s) ds.$$

Then the problem (3) admits a solution u satisfying

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{M(\int_r^R A(s) ds)^{-\alpha}} = 1,$$

where $d(x) := \text{dist}(x, \partial B_R(0))$ and

$$M = [\alpha(\alpha + 1)A_0 - \alpha]^\alpha, \quad \alpha = \frac{m}{p - m}.$$

Here

$$A_0 = \lim_{r \rightarrow R} \frac{A(r)^2}{a(r) \int_r^R A(s) ds}.$$

Furthermore, if $\lambda \geq 0$, then (3) admits a unique solution.

Theorem 2 is a sharp improvement of M. Delgado et al. [4, Theorem 1.2].

In the following examples, we shall obtain the precise blow-up rate for some special cases and compare our results with others'.

Example 3. $a(r) = a_0 > 0$. An easy calculation shows that $A_0 = 2$ and therefore

$$u(x) \sim (2\alpha^2 + \alpha)^\alpha \left[\frac{1}{2} a_0 (R - r)^2 \right]^{-\alpha},$$

where $\alpha = \frac{p}{p-m}$. In particular, when $m = 1$, the result agrees with [2].

Example 4. $a(r) = a_0(R - r)^\nu$, then

$$A(r) = \frac{1}{\nu + 1} a_0 (R - r)^{\nu+1}, \quad A_0 = \frac{\nu + 2}{\nu + 1}, \quad M = \left[\alpha(\alpha + 1) \frac{\nu + 2}{\nu + 1} - \alpha \right]^\alpha.$$

Therefore,

$$u(x) \sim M \left[\frac{a_0}{(\nu + 1)(\nu + 2)} \right]^{-\alpha} (R - r)^{-\frac{m(\nu+2)}{p-m}}.$$

The result agrees with [4, Theorem 1.2].

Example 5. Here we show an example that covers a more general case which is not included in the previous results. Let $a(r) = a_0 \exp(-(R - r)^{-\nu})$, then $\lim_{r \rightarrow R} \frac{A(r)}{a(r)} = \frac{1}{\nu} (R - r)^{\nu+1} = 0$ for $\nu > 0$. Furthermore,

$$\left(\frac{A(r)}{a(r)} \right)' = -1 + \frac{A(r)}{a(r)} \nu (R - r)^{-\nu-1} \rightarrow 0$$

as $r \rightarrow R$. Thus $A_0 = 1$, $M = \alpha^{2\alpha}$ with $\alpha = \frac{m}{p-m}$. Therefore,

$$u(x) \sim M \left[\int_r^R A(s) ds \right]^{-\alpha}.$$

This implies that $u(x)$ goes to ∞ at the boundary faster than any power function.

We organize the paper as follows. In Section 2, we establish some preliminary results that will be used later to prove our main theorem. In Section 3, we study the exact blow-up rate and prove the uniqueness based on the blow-up rate.

2. Some preliminary results

In this section we include some useful preliminary results. The first one is an extension of [7, Lemma 4].

Theorem 6. Suppose \underline{u} and \bar{u} satisfy

$$-\Delta \underline{u} \leq \lambda \underline{u}^{1/m} - a(x) \underline{u}^{p/m} \quad \text{in } \Omega,$$

$$-\Delta \bar{u} \geq \lambda \bar{u}^{1/m} - a(x) \bar{u}^{p/m} \quad \text{in } \Omega,$$

$$\lim_{d(x) \rightarrow 0^+} \underline{u} = \infty, \quad \lim_{d(x) \rightarrow 0^+} \bar{u} = \infty$$

and

$$\underline{u} \leq \bar{u} \quad \text{in } \Omega.$$

Then (3) possesses a solution u satisfying $\underline{u} \leq u \leq \bar{u}$.

The proof of this theorem follows from [1, Theorem A]. We shall omit here.

We shall also need the following lemmas that provide us information on $a(x)$ as $d(x) \rightarrow 0^+$.

Lemma 7. *Let $f(r) \in C([0, R], [0, \infty))$ and $f(r) > 0$ for $r \in (0, R]$. We define*

$$F(r) = \int_0^r f(s) ds, \quad G(r) = \int_0^r \int_0^s f(t) dt ds.$$

If there exists $g \in C^1([0, R])$ such that

$$g(0) = 0, \quad g'(0) \geq 0, \quad \text{and} \quad \lim_{r \rightarrow 0^+} \frac{F(r)}{g(r)f(r)} = c > 0,$$

then we have

(a) $\lim_{r \rightarrow 0^+} \frac{F^\mu(r)}{f(r)} = 0, \quad \mu \geq 1;$

(b) $\lim_{r \rightarrow 0^+} \frac{G(r)}{F(r)} = 0;$

(c) $\lim_{r \rightarrow 0^+} \frac{F^2(r)}{G(r)f(r)} = C_0 \geq 1.$

Proof. Since

$$\frac{F^\mu(r)}{f(r)} = \frac{F(r)}{g(r)f(r)} g(r) F^{\mu-1}(r) \rightarrow 0,$$

statement (a) follows easily. We can also prove statement (b) using L'Hospital rule.

To prove statement (c), we note that

$$\begin{aligned} \lim_{r \rightarrow 0^+} \frac{F^2(r)}{G(r)f(r)} &= \lim_{r \rightarrow 0^+} \frac{F(r)}{g(r)f(r)} \frac{F(r)g(r)}{G(r)} = c \lim_{r \rightarrow 0^+} \frac{F(r)g'(r) + F'(r)g(r)}{F(r)} \\ &= c \left(g'(0) + \frac{1}{c} \right) = 1 + cg'(0) \geq 1. \quad \square \end{aligned}$$

An immediate consequence of Lemma 7 is the following:

Lemma 8. *Let $a(r) \in C([0, R]; [0, \infty))$ and $A(r) = \int_r^R a(s) ds$. If $\frac{A(r)}{a(r)}$ is differentiable in $[0, R]$ and $\lim_{r \rightarrow R} \frac{A(r)}{a(r)} = 0, \lim_{r \rightarrow R} \left[\frac{A(r)}{a(r)} \right]' \leq 0$, then we have*

$$\lim_{r \rightarrow R} \frac{A^\mu(r)}{a(r)} = 0, \quad \mu \geq 1, \quad \lim_{r \rightarrow R} \frac{\int_r^R A(s) ds}{A(r)} = 0,$$

and

$$\lim_{r \rightarrow R} \frac{A(r)^2}{a(r) \int_r^R A(s) ds} = A_0 \geq 1.$$

3. Proof of Theorem 2

Consider the following singular problem

$$\begin{cases} -\phi'' - \frac{N-1}{r}\phi' = \lambda\phi^{1/m} - a(r)\phi^{p/m} & \text{in } (0, R), \\ \lim_{r \rightarrow R} \phi(r) = \infty, \\ \phi'(0) = 0, \end{cases} \tag{4}$$

where $R > 0$, $\lambda \in \mathbb{R}$ and $a \in C([0, R]; [0, \infty))$.

We claim that for each $\epsilon > 0$, the problem possesses a positive solution ϕ_ϵ such that

$$1 - \epsilon \leq \liminf_{r \rightarrow R} \frac{\phi_\epsilon(r)}{M(\int_r^R A(s) ds)^{-\alpha}} \leq \limsup_{r \rightarrow R} \frac{\phi_\epsilon(r)}{M(\int_r^R A(s) ds)^{-\alpha}} \leq 1 + \epsilon, \tag{5}$$

where

$$\alpha = \frac{m}{p-m}, \quad M = [\alpha(\alpha+1)A_0 - \alpha]^\alpha,$$

and

$$A_0 = \lim_{r \rightarrow R} \frac{(\int_r^R a(s) ds)^2}{a(r) \int_r^R \int_s^R a(t) dt ds}.$$

Thus the function

$$u_\epsilon(x) := \phi_\epsilon(r), \quad r := \|x\|,$$

provides us with a radially symmetric positive large solution of

$$\begin{cases} -\Delta u = \lambda u^{1/m} - a(r)u^{p/m} & \text{in } B_R(0), \\ u = \infty & \text{on } \partial B_R(0), \end{cases} \tag{6}$$

satisfying

$$1 - \epsilon \leq \liminf_{d(x) \rightarrow 0} \frac{u_\epsilon(x)}{M(\int_r^R A(s) ds)^{-\alpha}} \leq \limsup_{d(x) \rightarrow 0} \frac{u_\epsilon(x)}{M(\int_r^R A(s) ds)^{-\alpha}} \leq 1 + \epsilon,$$

where $d(x) := \text{dist}(x, \partial B_R(0)) = R - r$.

We prove the claim by constructing a supersolution and a subsolution with the same blow-up rate.

First, we claim that, for each $\epsilon > 0$, there exists a constant $A_\epsilon > 0$ such that for all $A_+ > A_\epsilon$,

$$\bar{\phi}_\epsilon(r) = A_+ + B_+ \left(\frac{r}{R}\right)^2 \left[\int_r^R A(s) ds \right]^{-\alpha},$$

where

$$\alpha = \frac{m}{p-m}, \quad B_+ = (1 + \epsilon)[\alpha(\alpha+1)A_0 - \alpha]^\alpha$$

is a positive supersolution of (4).

For simplicity, we denote

$$A^*(r) := \int_r^R A(s) ds = \int_r^R \int_s^R a(t) dt ds.$$

An easy calculation shows

$$\begin{aligned} \bar{\phi}'_\epsilon(r) &= 2B_+ \frac{r}{R^2} [A^*(r)]^{-\alpha} - \alpha B_+ \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha-1} [A^*(r)]', \\ \bar{\phi}''_\epsilon(r) &= 2B_+ \frac{1}{R^2} [A^*(r)]^{-\alpha} - 4\alpha B_+ \frac{r}{R^2} [A^*(r)]^{-\alpha-1} [A^*(r)]' \\ &\quad + \alpha(\alpha + 1)B_+ \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha-2} [A^*(r)']^2 \\ &\quad - \alpha B_+ \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha-1} [A^*(r)]''. \end{aligned}$$

By the assumptions of the theorem, $\lim_{r \rightarrow R} A^*(r) = 0$, we have

$$\lim_{r \rightarrow R} \bar{\phi}_\epsilon(r) = \infty$$

and

$$\bar{\phi}'_\epsilon(0) = 0.$$

Thus in order to show $\bar{\phi}_\epsilon$ is a supersolution, we only need to show

$$\begin{aligned} &-2N \frac{B_+}{R^2} [A^*(r)]^{-\alpha} + (N + 3)\alpha B_+ \frac{r}{R^2} [A^*(r)]^{-\alpha-1} [A^*(r)]' \\ &- \alpha(\alpha + 1)B_+ \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha-2} [A^*(r)']^2 + \alpha B_+ \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha-1} [A^*(r)]'' \\ &\geq \lambda [A^*(r)]^{-\alpha/m} \left[A_+ A^*(r)^\alpha + B_+ \left(\frac{r}{R}\right)^2 \right]^{1/m} \\ &\quad - a(r) [A^*(r)]^{-p\alpha/m} \left[A_+ A^*(r)^\alpha + B_+ \left(\frac{r}{R}\right)^2 \right]^{p/m}. \end{aligned} \tag{7}$$

Multiplying on both sides of inequality (7) by $\frac{1}{a(r)} [A^*(r)]^{p\alpha/m}$ and taking into account that

$$\alpha = \frac{m}{p - m}, \quad \text{i.e.,} \quad \frac{p\alpha}{m} - \alpha = 1,$$

we have

$$\begin{aligned} &-2N \frac{B_+}{R^2} \frac{A^*(r)}{a(r)} + (N + 3)\alpha B_+ \frac{r}{R^2} \frac{[A^*(r)]'}{a(r)} \\ &- \alpha(\alpha + 1)B_+ \left(\frac{r}{R}\right)^2 \frac{[A^*(r)']^2}{A^*(r)a(r)} + \alpha B_+ \left(\frac{r}{R}\right)^2 \frac{[A^*(r)]''}{a(r)} \\ &\geq \lambda \frac{1}{a(r)} [A^*(r)]^{\frac{p-1}{p-m}} \left[A_+ A^*(r)^\alpha + B_+ \left(\frac{r}{R}\right)^2 \right]^{1/m} - \left[A_+ A^*(r)^\alpha + B_+ \left(\frac{r}{R}\right)^2 \right]^{p/m}. \end{aligned} \tag{8}$$

By Lemma 8, we have

$$\lim_{r \rightarrow R} \frac{A^*(r)}{a(r)} = 0, \quad \lim_{r \rightarrow R} \frac{[A^*(r)]'}{a(r)} = 0, \quad \lim_{r \rightarrow R} \frac{[A^*(r)']^2}{A^*(r)a(r)} = A_0$$

and

$$\lim_{r \rightarrow R} \frac{[A^*(r)]^\mu}{a(r)} = \lim_{r \rightarrow R} \frac{[A^*(r)]^\mu A^\mu(r)}{A^\mu(r) a(r)} = 0,$$

where $\mu = \frac{p-1}{p-m} \geq 1$.

Thus at $r = R$, inequality (8) becomes

$$-\alpha(\alpha + 1)B_+ A_0 + \alpha B_+ \geq -B_+^{p/m}. \tag{9}$$

Therefore, by making the following choice

$$B_+ = (1 + \epsilon)[\alpha(\alpha + 1)A_0 - \alpha]^\alpha,$$

the inequality (8) is satisfied in $(R - \delta, R]$ for some $\delta = \delta(\epsilon) > 0$. Finally, by choosing a sufficiently large $A_+ \geq A_\epsilon$, (8) is satisfied in the whole interval $[0, R]$ since $p > m \geq 1$ and $A^*(r)$ is bounded away from zero in $[0, R - \delta]$. This concludes the claim.

Next we construct a subsolution with the same blow-up rate as the supersolution constructed above. In fact, for each sufficiently small $\epsilon > 0$, there exists $A_- < 0$ for which the function

$$\underline{\phi}_\epsilon(r) := \max \left\{ 0; A_- + B_- \left(\frac{r}{R} \right)^2 [A^*(r)]^{-\alpha} \right\}$$

provides us a nonnegative subsolution if

$$B_- = (1 - \epsilon)[\alpha(\alpha + 1)A_0 - \alpha]^\alpha,$$

where $\alpha = \frac{m}{p-m}$.

Indeed, it is easy to see that $\underline{\phi}_\epsilon$ is a subsolution if in the region where

$$A_- + B_- \left(\frac{r}{R} \right)^2 [A^*(r)]^{-\alpha} \geq 0$$

the following inequality is satisfied:

$$\begin{aligned} & -2N \frac{B_- A^*(r)}{R^2 a(r)} + (N + 3)\alpha B_- \frac{r [A^*(r)]'}{R^2 a(r)} \\ & -\alpha(\alpha + 1)B_- \left(\frac{r}{R} \right)^2 \frac{[A^*(r)']^2}{A^*(r)a(r)} + \alpha B_- \left(\frac{r}{R} \right)^2 \frac{[A^*(r)]''}{a(r)} \\ & \leq \lambda \frac{1}{a(r)} [A^*(r)]^{\frac{p-1}{p-m}} \left[A_- A^*(r)^\alpha + B_- \left(\frac{r}{R} \right)^2 \right]^{1/m} \left[A_- A^*(r)^\alpha + B_- \left(\frac{r}{R} \right)^2 \right]^{p/m}. \end{aligned} \tag{10}$$

At $r = R$, the above inequality is equivalent to

$$-\alpha(\alpha + 1)B_- A_0 + \alpha B_- \leq -B_-^{p/m},$$

i.e.,

$$B_- \leq [\alpha(\alpha + 1)A_0 - \alpha]^\alpha.$$

Thus there exists $\delta = \delta(\epsilon) > 0$ for which (10) is satisfied in $[R - \delta, R]$.

Moreover, for each $A_- < 0$, there exists a constant $z = z(A_-) \in (0, R)$ such that

$$A_- + B_- \left(\frac{r}{R} \right)^2 [A^*(r)]^{-\alpha} < 0 \quad \text{if } r \in [0, z(A_-)),$$

while

$$A_- + B_- \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha} \geq 0 \quad \text{if } r \in [z(A_-), R].$$

The claim above follows from the observation below

$$\lim_{r \rightarrow R} A_- + B_- \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha} = \infty,$$

$$\lim_{r \rightarrow 0} A_- + B_- \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha} = A_- < 0$$

and

$$\left[A_- + B_- \left(\frac{r}{R}\right)^2 [A^*(r)]^{-\alpha} \right]' < 0 \quad \text{in } (0, R),$$

where $'$ represents the derivative with respect to r .

In fact, $z(A_-)$ is decreasing in A_- and

$$\lim_{A_- \rightarrow -\infty} z(A_-) = R, \quad \lim_{A_- \rightarrow 0^-} z(A_-) = 0.$$

Then by choosing A_- such that $z(A_-) = R - \delta$, it follows that $\underline{\phi}_\epsilon$ provides us a subsolution of (4).

Finally, since

$$1 - \epsilon \leq \liminf_{r \rightarrow R} \frac{\underline{\phi}_\epsilon(r)}{M(\int_r^R A(s) ds)^{-\alpha}} \leq \limsup_{r \rightarrow R} \frac{\bar{\phi}_\epsilon(r)}{M(\int_r^R A(s) ds)^{-\alpha}} \leq 1 + \epsilon,$$

it follows from Theorem 6 that there exists a solution of (4), denoted by ϕ_ϵ , satisfying (5). Letting $\epsilon \rightarrow 0^+$ we obtain

$$\lim_{r \rightarrow R} \frac{\phi(r)}{M(\int_r^R A(s) ds)^{-\alpha}} = 1.$$

For any two arbitrary large solutions $u_1(x) = \phi_1(r)$, $u_2(x) = \phi_2(r)$, it readily follows that

$$\lim_{d(x) \rightarrow 0^+} \frac{u_1(x)}{u_2(x)} = 1.$$

Hence for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that

$$(1 - \epsilon)u_2 \leq u_1 \leq (1 + \epsilon)u_2$$

for any $x \in \Omega$ with $0 < d(x) \leq \delta$.

It is clear that u_1 is a positive solution of the boundary value problem

$$\begin{cases} -\Delta\phi = \lambda\phi^{1/m} - a(x)\phi^{p/m} & \text{in } \Omega_\delta, \\ \phi = u_1 & \text{on } \partial\Omega_\delta, \end{cases} \tag{11}$$

where $\Omega_\delta := \{x \in \Omega \mid d(x) > \delta\}$. It is easy to see that $\phi^- = (1 - \epsilon)u_2$ provides us with a positive sub-solution and $\phi^+ = (1 + \epsilon)u_2$ provides us with a super-solution of (4). Thus the unique solution of (4) u_1 satisfies $\phi^- \leq u_1 \leq \phi^+$ in Ω_δ . Thus

$$(1 - \epsilon)u_2 \leq u_1 \leq (1 + \epsilon)u_2$$

for any $x \in \Omega$. Passing to the limit $\epsilon \rightarrow 0^+$, we conclude $u_1 \equiv u_2$.

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