Global and blow-up solutions for a mutualistic model

Peng Feng *

Department of Physical Sciences and Mathematics, Florida Gulf Coast University, Fort Myers, FL 33965, United States

Received 7 September 2006; accepted 16 January 2007

Abstract

We study the global and blow-up solutions for a strong degenerate reaction–diffusion system modeling the interactions of two biological species. The local existence and uniqueness of a classical solution are established. We further give the critical exponent for reaction and absorption terms for the existence of global and blow-up solutions. We show that the solution may blow up if the intraspecific competition is weak. This supports ecologist A.J. Nicholson’s conclusion that intraspecific competition is the main factor regulating population size.

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Keywords: Mutualistic model; Degenerate reaction–diffusion system; Global solution; Blow-up solution

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the following nonlinear parabolic system:

$$
\begin{align*}
    u_t &= u^p \Delta u + u^{p+1}(a_1 - b_1 u^{r_i} + c_1 v^l) \quad \text{in } \Omega \times \mathbb{R}^+,
    \\
    v_t &= v^q \Delta v + v^{q+1}(a_2 + b_2 u^s - c_2 v^m) \quad \text{in } \Omega \times \mathbb{R}^+,
    \\
    u(x, t) &= v(x, t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+,
    \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x) \quad \text{for } x \in \Omega,
\end{align*}
$$

where $p, q, r, s \geq 1, l \geq 1, m, a_i, b_i, c_i (i = 1, 2)$ are positive constants. The initial data $u_0(x)$ and $v_0(x)$ satisfy

$$
\begin{align*}
    u_0(x), v_0(x) &\in C^1(\bar{\Omega}),
    \\
    u_0(x), v_0(x) &> 0 \quad \text{in } \Omega,
    \\
    u_0(x) = v_0(x) &= 0, \quad \frac{\partial u_0}{\partial \eta} < 0, \quad \frac{\partial v_0}{\partial \eta} < 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

Here $\eta$ is the outward normal vector on $\partial \Omega$.

We call $(u, v)$ a classical solution of (1) if $(u, v) \in [C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))]^2$ for some $0 < T \leq \infty$ and $(u, v)$ satisfies the differential equations in (1) and the initial and boundary conditions.

System (1) is usually referred as the cooperative two-species Lotka–Volterra model. It provides a simple model for describing the interaction of two diffusive biological species. The unknown functions $u$ and $v$ represent the densities...
of two species, \(a_1\) and \(a_2\) are the growth rates. The reaction terms \(v^f\) and \(u^g\) represent the assumption that each species finds its subsistence from the activity of the other one, or interspecific competition. The absorption terms \(u^e\) and \(v^m\) represent the competition among the same species, or intraspecific competition.

In [7], Pao studied the following mutualistic model:

\[
\begin{aligned}
& u_t = d_1 \Delta u + u(a_1 - b_1 u + c_1 v) \\
& v_t = d_2 \Delta v + v(a_2 + b_2 u - c_2 v) \\
& u(x, t) = v(x, t) = 0 \\
& u(x, 0) = u_0(x), \\& v(x, 0) = v_0(x) 
\end{aligned}
\]  

In Section 2.1, Local existence

He showed that the solution of (3) is unique and global when \(b_2c_1 < b_1c_2\); the solution blows up for any \(a_1 \geq 0, a_2 \geq 0\) with suitable initial data when \(b_2c_1 > b_1c_2\). For the critical case \(b_2c_1 = b_1c_2\), he showed that the solution blows up in finite time for large \(a_1\) and \(a_2\). These results imply that the solution is global if the intraspecific competition is strong, while the solution may blow up if the intraspecific competition is weak.

For the following similar system:

\[
\begin{aligned}
& u_t = u^p(\Delta u + av) \\
& v_t = v^q(\Delta v + bu)
\end{aligned}
\]  

Wang [8] proved that the solution \((u, v)\) exists globally if and only if \(ab \leq \lambda_1^2\), where \(\lambda_1\) is the first eigenvalue of \(-\Delta\) in \(\Omega\) with homogeneous Dirichlet boundary condition. For similar systems that have been studied, we refer the readers to [2–6,9,10].

When \(p = q, a_1 = a_2, c_i = 0\) and \(u_0(x) = v_0(x)\), system (1) is then reduced to a single initial boundary value problem:

\[
\begin{aligned}
& u_t = u^p(\Delta u + a_1 u) \\
& u(x, t) = 0 \\
& u(x, 0) = u_0(x), \\& x \in \Omega
\end{aligned}
\]  

This problem has been discussed by many authors. For example, for when \(p = 2\), Friedman and Mcleod [2] proved that if \(\lambda_1 > a_1\) then the solution exists globally but it blows up in finite time for \(\lambda > a_1\). For other related results, see for example [10] and references therein. For a survey of blowup phenomena in parabolic equations, we refer the readers to [1].

A key feature of (1) is its degeneracy, since \(u = v = 0\) on \(\partial \Omega\). Our main goal in this paper is to establish the local existence as well as the global existence and nonexistence of the solutions. Our main results are stated in the following theorems.

**Theorem 1.** If \(ls < rm\), then all solutions of (1) are global and uniformly bounded.

**Theorem 2.** If \(ls = rm\), then:

1. (1) has a unique global solution \((u, v)\) which is uniformly bounded for \(b_1^e c_2^e > b_2^e c_1^e\), i.e., \(b_1^m c_2^m > b_2^m c_1^m\).
2. The solution of (1) blows up in finite time for \(b_2^e c_1^e \leq b_1^e c_2^e\) provided that \(\min[a_1, a_2] > \lambda_1\).

**Theorem 3.** If \(ls > rm\), then the solution of (1) blows up in finite time for \(b_1^e c_2^e \leq b_2^e c_1^e\) provided that \(\min[a_1 - b_2, a_2] > \lambda_1\) or \(\min[a_1, a_2 - b_2] > \lambda_1\).

This paper is organized as follows. In the next section, we establish the local existence, uniqueness and give several comparison principles. In Section 3, we prove Theorems 1–3.

2. **Local existence, uniqueness and comparison principle**

2.1. **Local existence**

Since \(u = v = 0\) on the boundary \(\partial \Omega\), the equations in (1) are not strictly parabolic. The standard parabolic theory cannot be used to prove the existence of a solution directly. To prove local existence, we modify the boundary
conditions. We consider the following regularized system:

\[
\begin{align*}
    u_{\epsilon} &= f_\epsilon(u_{\epsilon}) \Delta u_{\epsilon} + f_\epsilon(u_{\epsilon})u_{\epsilon}(a_1 - b_1 u_{\epsilon}^p + c_1 v_{\epsilon}^q) \quad \text{in } \Omega \times \mathbb{R}_+, \\
    v_{\epsilon} &= g_\epsilon(v_{\epsilon}) \Delta v_{\epsilon} + g_\epsilon(v_{\epsilon})v_{\epsilon}(a_2 + b_2 u_{\epsilon}^p - c_v v_{\epsilon}^m) \quad \text{in } \Omega \times \mathbb{R}_+, \\
    u_{\epsilon}(x, t) &= v_{\epsilon}(x, t) = \epsilon \quad \text{on } \partial \Omega \times \mathbb{R}_+, \\
    u_{\epsilon}(x, 0) &= u_0(x) + \epsilon, \quad v_{\epsilon}(x, 0) = v_0(x) + \epsilon \quad \text{for } x \in \bar{\Omega},
\end{align*}
\]

(6)

where \( f_\epsilon \) and \( g_\epsilon \) are smooth and positive functions satisfying

\[
    f_\epsilon(u) = \begin{cases} 
    u^p, & u \geq \epsilon, \\
    \epsilon^p, & u < \epsilon, 
    \end{cases}
    \quad \text{and} \quad 
    g_\epsilon(v) = \begin{cases} 
    v^q, & v \geq \epsilon, \\
    \epsilon^q, & v < \epsilon. 
    \end{cases}
\]

(7)

By standard parabolic theory, we get the following lemma:

**Lemma 4.** There exists a unique solution \( u_\epsilon, v_\epsilon \in C(\bar{\Omega} \times [0, T(\epsilon)]) \cap C^{2,1}(\Omega \times (0, T(\epsilon))) \) for system (6) such that \( u_\epsilon \geq 0 \) and \( v_\epsilon \geq 0 \), where \( 0 < T(\epsilon) \leq \infty \).

The following proposition provides a lower bound of the solution for system (6).

**Proposition 5.** If \( \epsilon \leq \min\{(a_1/b_1)^{1/r}, (a_2/c_2)^{1/m}\} \), then \( u_\epsilon \geq \epsilon \) and \( v_\epsilon \geq \epsilon \) in \( \bar{\Omega} \times [0, T(\epsilon)) \).

To prove this proposition, we need the following lemmas.

**Lemma 6.** Let \( z_i \in C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)) \) \((i = 1, 2)\) and satisfy

\[
\begin{align*}
    z_{it} - a_i(x, t) \Delta z_i &\geq \sum_{j=1}^{2} b_{ij}(x, t) z_j \quad \text{in } \Omega \times (0, T), \\
    z_i(x, t) &\geq 0 \quad \text{on } \partial \Omega \times (0, T), \\
    z_i(x, t) &\geq 0 \quad \text{for } x \in \bar{\Omega}.
\end{align*}
\]

(8)

If \( a_i, b_{ij} \in C(\bar{\Omega} \times [0, T)) \) such that \( a_i > 0 \) and \( b_{ij} \geq 0 \) for \( i \neq j \) in \( \Omega \times (0, T) \), then \( z_i \geq 0 \) in \( \bar{\Omega} \times [0, T) \).

**Lemma 7.** Let \( z_i \in C(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)) \) \((i = 1, 2)\) and satisfy

\[
\begin{align*}
    z_{it} - a_i(x, t) \Delta z_i &= P_i(z_1, z_2) \quad \text{in } \Omega \times (0, T), \\
    z_i(x, t) &\geq 0 \quad \text{on } \partial \Omega \times (0, T), \\
    z_i(x, t) &\geq 0 \quad \text{for } x \in \bar{\Omega}.
\end{align*}
\]

(9)

If \( a_i \in C(\bar{\Omega} \times [0, T)) \) such that \( a_i > 0 \) and \( P_i(z_1, z_2) \) is Lipschitz continuous with respect to \( z_i \) and it holds that \( P_i(z_1, z_2) \geq 0 \) for \( (z_1, z_2) \in \mathbb{R}_+^2 \) whenever \( z_i = 0 \), then \( z_i \geq 0 \) for all \( x \in \bar{\Omega} \).

The proof of Lemma 6 can be found in [7]. Here we prove Lemma 7.

**Proof.** We introduce the following notation:

\[
\begin{align*}
    Z &= (z_1, z_2), \quad Z_1 = (0, z_2), \quad Z_2 = (z_1, 0).
\end{align*}
\]

Observe that \( P_i(Z) \) can be rewritten as

\[
P_i(Z) = P_i(Z) - P_i(Z_i) + P_i(Z_i) \geq P_i(Z) - P_i(Z_i)
\]

since \( P_i(Z_i) \geq 0 \).

By the Lipschitz continuity of \( P_i \) with respect to \( z_i \), there exists \( 0 \leq \tilde{z}_i \leq z_i \) such that

\[
P_i(Z) - P_i(Z_i) = \left. \frac{\partial P_i(Z)}{\partial Z_i} \right|_{Z_i=\tilde{z}_i} z_i.
\]

Moreover, the Lipschitz continuity guarantees that for \( t \in [0, T) \), there exists a positive constant \( C \) such that

\[
\left. \frac{\partial P_i(Z)}{\partial Z_i} \right|_{Z_i=\tilde{z}_i} \leq C.
\]

Please cite this article in press as: P. Feng, Global and blow-up solutions for a mutualistic model, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.01.060
It follows that

\[ P_i(Z) \geq -C \tilde{z}_i \quad \text{for } t \in [0, T). \]

Applying Lemma 6, we have \( \tilde{z}_i(x, t) \geq 0 \) in \( \tilde{\Omega} \times [0, T) \). \( \square \)

We are now in position to prove Proposition 5.

**Proof.** Let \( z_1 = u_\epsilon - \epsilon, z_2 = v_\epsilon - \epsilon \); we have

\[
\begin{align*}
\begin{cases}
\tilde{z}_{1t} = \tilde{f}_e(z_1) \left[ \Delta \tilde{z}_1 + (a_1 - b_1(z_1 + \epsilon)^r)z_1 + \epsilon \left( a_1 - b_1(z_1 + \epsilon)^r \right) \right] + c_1 \tilde{f}_e(z_1) (z_1 + \epsilon)(z_2 + \epsilon)^l \\
in \tilde{\Omega} \times (0, T(\epsilon)),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\tilde{z}_{2t} = \tilde{g}_e(z_2) \left[ \Delta \tilde{z}_2 + (a_2 - c_2(z_2 + \epsilon)^m)z_2 + \epsilon \left( a_2 - c_2(z_2 + \epsilon)^m \right) \right] + b_2 \tilde{g}_e(z_2) (z_2 + \epsilon)(z_1 + \epsilon)^l \\
in \tilde{\Omega} \times (0, T(\epsilon)),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
z_1(x, t) = z_2(x, t) = 0 \quad \text{on } \partial \tilde{\Omega} \times (0, T(\epsilon)),
\end{align*}
\]

\[
\begin{align*}
z_1(x, 0) = u_0(x), \quad z_2(x, 0) = v_0(x) \quad \text{for } x \in \tilde{\Omega},
\end{align*}
\]

where

\[
\tilde{f}_e(z_1) = \begin{cases}
(z_1 + \epsilon)^p, & z_1 \geq 0, \\
\epsilon^p, & z_1 < 0,
\end{cases} \quad \text{and} \quad \tilde{g}_e(z_2) = \begin{cases}
(z_2 + \epsilon)^q, & z_2 \geq 0, \\
\epsilon^q, & z_2 < 0.
\end{cases}
\]

Let

\[
P_1 = \tilde{f}_e(z_1) \left[ (a_1 - b_1(z_1 + \epsilon)^r)z_1 + \epsilon \left( a_1 - b_1(z_1 + \epsilon)^r \right) + c_1 (z_1 + \epsilon)(z_2 + \epsilon)^l \right],
\]

\[
P_2 = \tilde{g}_e(z_2) \left[ (a_2 - c_2(z_2 + \epsilon)^m)z_2 + \epsilon \left( a_2 - c_2(z_2 + \epsilon)^m \right) + b_2 (z_2 + \epsilon)(z_1 + \epsilon)^l \right].
\]

We can show that \( P_1 \) and \( P_2 \) satisfy the all conditions in Lemma 7 provided that \( \epsilon \leq \min(\{a_1b_1\}^{1/r}, \{a_2/c_2\}^{1/m}) \). Thus \( u_\epsilon \geq \epsilon \) and \( v_\epsilon \geq \epsilon \) in \( \tilde{\Omega} \times [0, T(\epsilon)) \). This ends the proof of this proposition. \( \square \)

Now that we have \( \epsilon \leq \min(\{a_1/b_1\}^{1/r}, \{a_2/c_2\}^{1/m}) \), \( u_\epsilon \geq \epsilon \) and \( v_\epsilon \geq \epsilon \) which gives \( f_e(u_\epsilon) = u_\epsilon^p \) and \( g_e(v_\epsilon) = v_\epsilon^q \) and hence \( (u_\epsilon, v_\epsilon) \) solves the following problem:

\[
\begin{align*}
\begin{cases}
u_{\epsilon t} = \nu_\epsilon^p \Delta \nu_\epsilon + \nu_\epsilon^{p+1}(a_1 - b_1\nu_\epsilon^r + c_1\nu_\epsilon^l) & \text{in } \tilde{\Omega} \times (0, T(\epsilon)), \\
u_{\epsilon t} = \nu_\epsilon^q \Delta \nu_\epsilon + \nu_\epsilon^{q+1}(a_2 - c_2\nu_\epsilon^m - c_2\nu_\epsilon^m) & \text{in } \tilde{\Omega} \times (0, T(\epsilon)), \\
u_\epsilon(x, t) = \epsilon & \text{on } \partial \tilde{\Omega} \times (0, T(\epsilon)), \\
u_\epsilon(x, 0) = u_0(x) + \epsilon & \text{for } x \in \tilde{\Omega}.
\end{cases}
\end{align*}
\]

Problem (11) is not degenerate since \( u_\epsilon, v_\epsilon \geq \epsilon \) for \( \epsilon \) sufficiently small. Since (11) is quasimonotone nondecreasing, we can apply the comparison principle (see chapter IV, 32 of [11]) and we have the following

**Proposition 8.** Let \( u_\epsilon, v_\epsilon \in C(\tilde{\Omega} \times [0, T(\epsilon))) \cap C^2(\tilde{\Omega} \times (0, T(\epsilon))) \) and satisfy

\[
\begin{align*}
\begin{cases}
u_{\epsilon t} \leq \nu_\epsilon^p \Delta \nu_\epsilon + \nu_\epsilon^{p+1}(a_1 - b_1\nu_\epsilon^r + c_1\nu_\epsilon^l) & \text{in } \tilde{\Omega} \times (0, T(\epsilon)), \\
u_{\epsilon t} \leq \nu_\epsilon^q \Delta \nu_\epsilon + \nu_\epsilon^{q+1}(a_2 - c_2\nu_\epsilon^m - c_2\nu_\epsilon^m) & \text{in } \tilde{\Omega} \times (0, T(\epsilon)), \\
u_\epsilon(x, t) \leq \epsilon & \text{on } \partial \tilde{\Omega} \times (0, T(\epsilon)), \\
u_\epsilon(x, 0) \leq u_0(x) + \epsilon & \text{for } x \in \tilde{\Omega}.
\end{cases}
\end{align*}
\]

Then \( (u_\epsilon, v_\epsilon) \) is \( \leq (u_\epsilon, v_\epsilon) \) on \( \tilde{\Omega} \times [0, T(\epsilon)). \)

Next we shall establish the following proposition:

**Proposition 9.** If \( \epsilon_1 < \epsilon_2 \leq \min(\{a_1/b_1\}^{1/r}, \{a_2/c_2\}^{1/m}) \), then \( T(\epsilon_1) \geq T(\epsilon_2) \) and \( (u_\epsilon, v_\epsilon) \leq (u_\epsilon, v_\epsilon) \) on \( \tilde{\Omega} \times [0, T(\epsilon_2)). \)

**Proof.** To prove this proposition, we let \( W_1 = u_\epsilon - u_\epsilon \) and \( W_2 = v_\epsilon - v_\epsilon \). From (11), we have

\[
\begin{align*}
W_{1t} &= u_\epsilon^p \Delta W_1 + b_1 W_1 + b_1 W_1 \quad \text{in } \tilde{\Omega} \times (0, \min(T(\epsilon_1), T(\epsilon_2))), \\
W_{2t} &= v_\epsilon^q \Delta W_2 + b_2 W_1 + b_2 W_2 \quad \text{in } \tilde{\Omega} \times (0, \min(T(\epsilon_1), T(\epsilon_2))), \\
W_1(x, t) &= W_2(x, t) = \epsilon_2 - \epsilon_1 > 0 \quad \text{on } \partial \tilde{\Omega} \times (0, \min(T(\epsilon_1), T(\epsilon_2))), \\
W_1(x, 0) &= W_2(x, 0) = \epsilon_2 - \epsilon_1 > 0 \quad \text{for } x \in \tilde{\Omega}.
\end{align*}
\]
where

\[
b_{11} = \Delta u_{\epsilon} p \int_0^1 [u_{\epsilon i} + s(u_{\epsilon 2} - u_{\epsilon i})]^{\rho - 1} ds + (a_1 + c_1 v'_{\epsilon i})(p + 1) \int_0^1 [u_{\epsilon i} + s(u_{\epsilon 2} - u_{\epsilon i})]^\rho ds \\
- b_1 (p + r + 1) \int_0^1 [u_{\epsilon i} + s(u_{\epsilon 2} - u_{\epsilon i})]^{\rho + r} ds,
\]

\[
b_{12} = c_1 u_{\epsilon 2}^{p + 1} l \int_0^1 [v_{\epsilon i} + s(v_{\epsilon 2} - v_{\epsilon i})]^{l - 1} ds \geq 0,
\]

\[
b_{21} = b_2 v_{\epsilon 2}^{q + 1} s \int_0^1 [u_{\epsilon i} + s(u_{\epsilon 2} - u_{\epsilon i})]^{s - 1} ds \geq 0,
\]

\[
b_{22} = \Delta v_{\epsilon} q \int_0^1 [v_{\epsilon i} + s(v_{\epsilon 2} - v_{\epsilon i})]^{q - 1} ds + (a_2 + b_2 u_{\epsilon i}) (q + 1) \int_0^1 [v_{\epsilon i} + s(v_{\epsilon 2} - v_{\epsilon i})]^q ds \\
- c_2 (q + m + 1) \int_0^1 [v_{\epsilon i} + s(v_{\epsilon 2} - v_{\epsilon i})]^{q + m} ds.
\]

Note that Lemma 6 cannot provide us with the argument as we do not know whether \( b_{11} \) and \( b_{22} \) are continuous or not on \( \partial \Omega \). However, this continuity condition can be weakened with stronger boundary and initial conditions. With the aid of the following lemma, the conclusion in Proposition 9 follows. \(\Box\)

**Lemma 10.** Let \( z_i \in C(\hat{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)) \) \( (i = 1, 2) \) and satisfy

\[
\begin{align*}
\left\{ \begin{array}{ll}
z_{\epsilon i} - a_i(x, t) \Delta z_i & \geq \sum_{j=1}^2 b_{ij}(x, t) z_j & \text{in } \Omega \times (0, T), \\
z_i(x, t) & > 0 & \text{on } \partial \Omega \times (0, T), \\
z_i(x, t) & > 0 & \text{for } x \in \hat{\Omega}.
\end{array} \right.
\end{align*}
\]

If \( a_i, b_{ij} \in C(\Omega \times [0, T)) \) such that \( a_i > 0 \) and \( b_{ij} \geq 0 \) for \( i \neq j \) in \( \Omega \times (0, T) \), then \( z_i \geq 0 \) in \( \hat{\Omega} \times [0, T) \).

The proof of this lemma can be found in [4].

The following lemma provides us with a positive lower bound on \((u_\epsilon, v_\epsilon)\) which will be applied in the proof of local existence.

**Lemma 11.** Let \( \epsilon \leq \min\{(a_1/b_1)^{1/r}, (a_2/c_2)^{1/m}\} \), \((u_\epsilon, v_\epsilon)\) be the solution of (11) and the positive constant \( \rho \geq \max\{0, k^p (\lambda_1 - a_1 + b_1 k^p), k^q (\lambda_1 - a_2 + c_2 k^m)\} \) for some \( k > 0 \); then

\[
(u_\epsilon, v_\epsilon) \geq (k \Phi(x) e^{-\rho t}, k \Phi(x) e^{-\rho t}) \text{ in } \hat{\Omega} \times [0, T(\epsilon)).
\]

Here \( \Phi(x) \) is the eigenfunction corresponding to the first eigenvalue \( \lambda_1 \) of \(-\Delta \) on \( \Omega \) with homogeneous boundary condition. Moreover, \( \Phi(x) \) can be normalized so that \( \max_{\hat{\Omega}} \Phi(x) = 1, \lambda_1 > 0 \) and \( \partial \Phi/\partial \eta < 0 \) on \( \partial \Omega \).

**Proof.** Set \( u_\epsilon(x, t) = k \Phi(x) e^{-\rho t}, v_\epsilon(x, t) = k \Phi(x) e^{-\rho t} \) where \( k > 0 \) can be chosen so that \( k \Phi(x) \leq \min\{u_0(x), v_0(x)\} \). A direct calculation yields

\[
u_\epsilon^p [\Delta u_\epsilon + u_\epsilon (a_1 - b_1 u_\epsilon^r + c_1 u_\epsilon^s)] = (k \Phi(x) e^{-\rho t})^p k \Phi(x) e^{-\rho t} (-\lambda_1 + a_1 - b_1 k^p \Phi(x)^{r-1})
+ c_1 k^p \Phi(x)^{r-1} e^{-\rho t}.
\]

\[
u_\epsilon^q [\Delta v_\epsilon + v_\epsilon (a_2 + b_2 u_\epsilon^s - c_2 v_\epsilon^m)] = (k \Phi(x) e^{-\rho t})^q k \Phi(x) e^{-\rho t} (-\lambda_1 + a_2 + b_2 k^s \Phi(x)^{m-1})
- c_2 k^m \Phi(x)^{m-1} e^{-\rho t}.
\]

Thus by taking

\[
\rho \geq \max\left\{0, k^p (\lambda_1 - a_1 + b_1 k^p), k^q (\lambda_1 - a_2 + c_2 k^m)\right\},
\]

we obtain
\[ \begin{align*}
    u_t &\leq u^p \Delta u + u^{p+1}(a_1 - b_1 u^r + c_1 v^l) & \text{in } \Omega \times (0, T(\varepsilon)), \\
    v_t &\leq v^q \Delta v + v^{q+1}(a_2 + b_2 u^s - c_2 u^m) & \text{in } \Omega \times (0, T(\varepsilon)), \\
    u(x,t) &= v(x,t) & \text{on } \partial \Omega \times (0, T(\varepsilon)), \\
    u(x,0) &\leq u_0(x) + \epsilon, & v(x,0) &\leq v_0(x) + \epsilon & \text{for } x \in \Omega.
\end{align*} \]

Proposition 8 implies that \((u_\varepsilon, v_\varepsilon) \geq (k \Phi(x)e^{-\rho t}, k \Phi(x)e^{-\rho t}). \]

It follows from Proposition 9 that there exists \(T : 0 < T \leq \infty\) and \((u,v)\) which is defined on \(\hat{\Omega} \times [0, T)\) such that \(T(\varepsilon) \rightarrow T\) and \((u_\varepsilon, v_\varepsilon) \rightarrow (u,v)\) as \(\varepsilon \searrow 0\). By Lemma 11, \((u,v)\) is positive in \(\Omega \times (0, T)\). The standard local Schauder estimates imply that \([\Omega \times (0, T)]^2\) such that

\[ (u_\varepsilon, v_\varepsilon) \rightarrow (u,v) \in [C^{2+\alpha,1+\alpha/2}(\Omega \times (0, T))]^2 \]

for any \(\Omega_0 \subset \Omega\) and \(0 < t_0 < t_1 < T\). Therefore, \((u,v)\) satisfies (1) in \(\Omega \times (0, T)\).

The continuity of \((u,v)\) in \(\Omega \times [0, T)\) follows from Lemma 11, \(L^p\) theory and the imbedding theorem. Similar to the arguments in [2], we can prove that \((u,v)\) is continuous on \(\partial\Omega \times (0, T)\). Therefore, we have the following local existence theorem.

**Theorem 12.** Problem (1) admits a positive classical solution \((u,v)\) where \((u,v) \in \[C(\hat{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)]\] and \(0 < T \leq +\infty\).

2.2. Uniqueness

In this part we prove the uniqueness of the positive solution of (1) for \(l \geq 1\) and \(s \geq 1\). Suppose that \((u_1, v_1)\) is another positive classical solution of (1) in \(\Omega \times (0, T)\). By Proposition 8, \((u_1, v_1) \leq (u_\varepsilon, v_\varepsilon)\) and thus \((u_1, v_1) \leq (u,v)\).

Following [10], we define \(\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \delta\}\) for \(\delta > 0\) sufficiently small. Let \(\Phi_\delta(x) > 0\) be the eigenfunction corresponding to the first eigenvalue \(\lambda_\delta\) of \(-\Delta \Phi_\delta(x) = \lambda \Phi_\delta(x), x \in \Omega,\) with homogeneous boundary condition. Multiplying the first equation in (1) by \(\Phi_\delta/u^p\) and integrating by parts, we have

\[ \int_{\Omega_\delta} \xi(u) \Phi_\delta dx = \int_{\Omega_\delta} \xi(u_0) \Phi_\delta dx - \int_0^T \int_{\partial \Omega} u \frac{\partial \Phi_\delta}{\partial \eta} ds dt - \lambda_\delta \int_0^T \int_{\Omega_\delta} u \Phi_\delta dx dt + \int_0^T \int_{\Omega_\delta} u(a_1 - b_1 u^r + c_1 v^l) \Phi_\delta dx dt, \]

where

\[ \xi(u) = \begin{cases} 
    \frac{u^{-p}}{1-p} & p > 1, \\
    \ln u & p = 1.
\end{cases} \]

Similarly, we have

\[ \int_{\Omega_\delta} \xi(u_1) \Phi_\delta dx = \int_{\Omega_\delta} \xi(u_0) \Phi_\delta dx - \int_0^T \int_{\partial \Omega} u_1 \frac{\partial \Phi_\delta}{\partial \eta} ds dt - \lambda_\delta \int_0^T \int_{\Omega_\delta} u_1 \Phi_\delta dx dt + \int_0^T \int_{\Omega_\delta} u_1(a_1 - b_1 u_1^r + c_1 v_1^l) \Phi_\delta dx dt. \]

Therefore

\[ \int_{\Omega_\delta} [\xi(u) - \xi(u_1)] \Phi_\delta dx = -\int_0^T \int_{\partial \Omega} (u - u_1) \frac{\partial \Phi_\delta}{\partial \eta} ds dt - \lambda_\delta \int_0^T \int_{\Omega_\delta} (u - u_1) \Phi_\delta dx dt + a_1 \int_0^T \int_{\Omega_\delta} (u - u_1) \Phi_\delta dx dt + b_1 \int_0^T \int_{\Omega_\delta} (u_1^r - u_1^{r+1}) \Phi_\delta dx dt + c_1 \int_0^T \int_{\Omega_\delta} (uv^l - u_1 v_1^l) \Phi_\delta dx dt. \]
Since \( u \geq u_1 \) on \( \Omega_\delta \) and for \( l \geq 1 \),
\[
c_1 \int_0^t \int_{\Omega_\delta} (uv' - u_1v'_1) \Phi_3 \, dx \, dt = c_1 \int_0^t \int_{\Omega_\delta} (u - u_1)v' \Phi_3 \, dx \, dt + c_1 \int_0^t \int_{\Omega_\delta} u_1(v' - v'_1) \Phi_3 \, dx \, dt \\
\leq M_1 \int_0^t \int_{\Omega_\delta} (u - u_1) \Phi_3 \, dx \, dt + M_2 \int_0^t \int_{\Omega_\delta} (v - v_1) \Phi_3 \, dx \, dt,
\]
where \( M_1 \geq 0, M_2 \geq 0 \).

Thus
\[
\int_{\Omega_\delta} [\xi(u) - \xi(u_1)] \Phi_3 \, dx \leq - \int_0^t \int_{\partial\Omega} (u - u_1) \frac{\partial \Phi_3}{\partial \eta} \, ds \, dt + (-\lambda_\delta + a_1 + M_1) \int_0^t \int_{\Omega_\delta} (u - u_1) \Phi_3 \, dx \, dt \\
+ M_2 \int_0^t \int_{\Omega_\delta} (v - v_1) \Phi_3 \, dx \, dt.
\]

Similarly, we can multiply the second equation in (1) by \( \Phi_3/v^q \) and follow the process described above to obtain
\[
\int_{\Omega_\delta} [\chi(v) - \chi(v_1)] \Phi_3 \, dx \leq - \int_0^t \int_{\partial\Omega} (v - v_1) \frac{\partial \Phi_3}{\partial \eta} \, ds \, dt + (-\lambda_\delta + a_\delta + M_3) \int_0^t \int_{\Omega_\delta} (v - v_1) \Phi_3 \, dx \, dt \\
+ M_4 \int_0^t \int_{\Omega_\delta} (u - u_1) \Phi_3 \, dx \, dt.
\]
Here
\[
\chi(v) = \begin{cases} 
\frac{v^{1-q}}{1-q} & q > 1, \\
\ln v & q = 1,
\end{cases}
\]
and \( M_3 \geq 0, M_4 \geq 0, s \geq 1 \).

On the other hand, it is easy to verify that
\[
\int_{\Omega_\delta} [\xi(u) - \xi(u_1)] \Phi_3 \, dx \geq M^{-p} \int_{\Omega_\delta} (u - u_1) \Phi_3 \, dx
\]
and
\[
\int_{\Omega_\delta} [\chi(v) - \chi(v_1)] \Phi_3 \, dx \geq M^{-q} \int_{\Omega_\delta} (v - v_1) \Phi_3 \, dx,
\]
where \( M \geq \max_{\Omega \times [0,T]} (u + v) \) for any \( \tau < T \). It follows that
\[
\int_{\Omega_\delta} [(u - u_1) + (v - v_1)] \Phi_3 \, dx \leq -\tilde{M} \int_0^t \int_{\partial\Omega} [(u - u_1) + (v - v_1)] \frac{\partial \Phi_3}{\partial \eta} \, ds \, dt \\
+ M_5 \int_0^t \int_{\Omega_\delta} [(u - u_1) + (v - v_1)] \Phi_3 \, dx \, dt,
\]
where \( \tilde{M} = \max\{M^p, M^q\} \) and \( M_5 = M^p(\lambda_\delta + a_1 + M_1 + M_4) + M^q(\lambda_\delta + a_2 + M_2 + M_3) \). We apply Gronwall’s Lemma to (16) and then take the limit as \( \delta \rightarrow 0 \) and it follows that \( (u, v) \equiv (u_1, v_1) \). Thus we have proved

**Theorem 13.** If \( s, l \geq 1 \), then there exists a unique positive classical solution \( (u, v) \in [C(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T))]^2 \) where \( 0 < T \leq \infty \). Moreover, if \( T < \infty \), then \( \lim_{t \rightarrow T^-} \max_{x \in \Omega} u(x, t) = \lim_{t \rightarrow T^-} \max_{x \in \bar{\Omega}} v(x, t) = +\infty \).

Please cite this article in press as: P. Feng, Global and blow-up solutions for a mutualistic model, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.01.060
2.3. Comparison principles

**Proposition 14** (Comparison Principle 1). Let \( u, v \in C(\tilde{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T)) \) and satisfy

\[
\begin{align*}
  u_t &\leq u_\alpha u + u^{p+1}(a_1 - b_1 u^r + c_1 v^l) \quad \text{in } \Omega \times (0, T), \\
  v_t &\leq v^\beta v + v^{q+1}(a_2 + b_2 u^s - c_2 v^m) \quad \text{in } \Omega \times (0, T), \\
  u(x, t) &\leq 0, \quad v(x, t) \leq 0 \quad \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) &\leq u_0(x), \quad v(x, 0) \leq v_0(x) \quad \text{for } x \in \tilde{\Omega}.
\end{align*}
\]

Then \((u, v) \leq (u, v)\) on \(\tilde{\Omega} \times [0, T]\).

**Proof.** This follows directly from Proposition 8 and the uniqueness result since we have

\[
(u_\epsilon(x, t), v_\epsilon(x, t)) \to (u(x, t), v(x, t)), \quad (u_\epsilon(x, t), v_\epsilon(x, t)) \to (u(x, t), v(x, t))
\]
as \(\epsilon \to 0^+\). \(\square\)

**Proposition 15** (Comparison Principle 2). Let \( \tilde{u}, \tilde{v} \in C(\tilde{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T)) \) and satisfy

\[
\begin{align*}
  \tilde{u}_t &\geq \tilde{u}_\alpha u + \tilde{u}^{p+1}(a_1 - b_1 \tilde{u}^r + c_1 \tilde{v}^l) \quad \text{in } \Omega \times (0, T), \\
  \tilde{v}_t &\geq \tilde{v}^\beta \tilde{v} + \tilde{v}^{q+1}(a_2 + b_2 \tilde{u}^s - c_2 \tilde{v}^m) \quad \text{in } \Omega \times (0, T), \\
  \tilde{u}(x, t) &\geq 0, \quad \tilde{v}(x, t) \geq 0 \quad \text{on } \partial \Omega \times (0, T), \\
  \tilde{u}(x, 0) &\geq u_0(x), \quad \tilde{v}(x, 0) \geq v_0(x) \quad \text{for } x \in \tilde{\Omega}.
\end{align*}
\]

Then \((\tilde{u}, \tilde{v}) \geq (u, v)\) on \(\tilde{\Omega} \times [0, \tilde{T}]\).

**Proof.** Let \((\tilde{u}_\epsilon(x, t), \tilde{v}_\epsilon(x, t))\) be the corresponding solutions of (11) with the boundary conditions replaced by \((\tilde{u}_\epsilon(x, t), \tilde{v}_\epsilon(x, t)) \geq (\epsilon, \epsilon)\) and the initial conditions replaced by \((\tilde{u}_\epsilon(x, 0), \tilde{v}_\epsilon(x, 0)) \geq (u_0(x) + \epsilon, v_0(x) + \epsilon)\). Since problem (11) is quasimonotone increasing in \((u_\epsilon, v_\epsilon)\), applying the comparison principle for the parabolic system we have

\[
(\tilde{u}_\epsilon(x, t), \tilde{v}_\epsilon(x, t)) \geq (u_\epsilon(x, t), v_\epsilon(x, t)).
\]

Letting \(\epsilon \to 0^+\) and by the uniqueness result we have \((\tilde{u}(x, t), \tilde{v}(x, t)) \geq (u(x, t), v(x, t))\). \(\square\)

**Proposition 16** (Comparison Principle 3). Let \( \tilde{u}, \tilde{v} \in C(\tilde{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T)) \) and satisfy

\[
\begin{align*}
  \tilde{u}_t &\geq \tilde{u}_\alpha u + \tilde{u}^{p+1}(a_1 - b_1 \tilde{u}^r + c_1 \tilde{v}^l) \quad \text{in } \Omega \times (0, T), \\
  \tilde{v}_t &\geq \tilde{v}^\beta \tilde{v} + \tilde{v}^{q+1}(a_2 + b_2 \tilde{u}^s - c_2 \tilde{v}^m) \quad \text{in } \Omega \times (0, T), \\
  \tilde{u}(x, t) &> 0, \quad \tilde{v}(x, t) > 0 \quad \text{on } \partial \Omega \times (0, T), \\
  \tilde{u}(x, 0) &> u_0(x), \quad \tilde{v}(x, 0) > v_0(x) \quad \text{for } x \in \tilde{\Omega}.
\end{align*}
\]

Then \((\tilde{u}, \tilde{v}) \geq (u, v)\) on \(\tilde{\Omega} \times [0, \tilde{T}]\).

**Proof.** Let \((z_1(x, t), z_2(x, t)) = (\tilde{u}(x, t) - u(x, t), \tilde{v}(x, t) - v(x, t))\). This proposition follows directly from Lemma 10. \(\square\)

3. Global existence and nonexistence

**Proof (Proof of Theorem 1).** We construct a constant supersolution of (1). Let \((\tilde{u}, \tilde{v}) = (\eta_1, \eta_2)\) be such that \((u_0, v_0) \leq (\eta_1, \eta_2)\). To prove that \((\eta_1, \eta_2)\) is a supersolution, it suffices to verify

\[
\begin{align*}
  b_1 \eta_1^r &\geq a_1 + c_1 \eta_2^l, \quad c_2 \eta_2^m \geq a_2 + b_2 \eta_1^s.
\end{align*}
\]

(17)

We choose \((\eta_1, \eta_2)\) such that \(a_1 \leq c_1 \eta_2^l, a_2 \leq b_2 \eta_1^s\). Then (17) holds if we can verify

\[
\begin{align*}
  b_1 \eta_1^r &\geq 2c_1 \eta_2^l, \quad c_2 \eta_2^m \geq 2b_2 \eta_1^s.
\end{align*}
\]
which is equivalent to
\[
\left( \frac{2b_2}{c_2} \right)^{1/m} \eta_1^{s/m} \leq \eta_2 \leq \left( \frac{b_1}{2c_1} \right)^{1/l} \eta_1^{r/l}.
\]  
(18)

Since \( ls < rm \), (18) clearly holds for suitably large \( \eta_1 \). By virtue of the comparison principle, we conclude that \((u, v)\) is uniformly bounded. \( \square \)

**Proof** (Proof of Theorem 2(1)). Like in the proof of Theorem 1, we choose \((\eta_1, \eta_2)\) so that
\[
a_1 \leq \delta c_1 \eta_1^l, \quad a_2 \leq \delta b_2 \eta_1^r,
\]
where \( \delta > 0 \) is to be determined later.

To show \((\bar{u}, \bar{v}) = (\eta_1, \eta_2)\) is a supersolution, it suffices to show that
\[
b_1 \eta_1^l \geq (1 + \delta)c_1 \eta_1^l, \quad c_2 \eta_2^m \geq (1 + \delta)b_2 \eta_1^r,
\]
which is equivalent to
\[
\left( \frac{(1 + \delta)b_2}{c_2} \right)^{1/m} \eta_1^{s/m} \leq \eta_2 \leq \left( \frac{b_1}{(1 + \delta)c_1} \right)^{1/l} \eta_1^{r/l}.
\]
In view of \( ls = rm \), it suffices to find a suitable \( \delta \) so that
\[
(1 + \delta)^{1+m} \leq \frac{b_1^{m}c_1^{m}}{2c_1^{m}b_2^{m}}.
\]
By the assumption of the theorem, we can always choose \( \delta \) small such that the inequality above holds. This ends the proof of Theorem 2(1). \( \square \)

To prove Theorem 2(2), we need to establish the following lemmas.

**Lemma 17.** Under the assumptions of Theorem 2(2), i.e., \( ls = rm \) and \( \min\{a_1, a_2\} > \lambda_1 \), the solution of (1) satisfies \((u, v) \geq (k_1 \Phi, k_2 \Phi)\) for suitable constants \( k_1, k_2 > 0 \).

**Proof.** By the assumption (2), there exists a positive constant \( k \) such that
\[
(u_0, v_0) \geq (k_1 \Phi(x), k_2 \Phi(x)) \quad \text{for} \ x \in \bar{\Omega}.
\]

Let \((u, v) = (k_1 \Phi(x), k_2 \Phi(x))\) with positive constant \( k_1, k_2 \leq k \). A direct calculation yields
\[
\Delta u + u(a_1 - b_1 u^r + c_1 \Phi^l) = k_1 \Phi(-\lambda_1 + a_1 - b_1 k_1^s \Phi^r + c_1 k_1^l \Phi^l),
\]
\[
\Delta v + v(a_2 + b_2 u^s - c_2 \Phi^m) = k_2 \Phi(-\lambda_1 + a_2 + b_2 k_2^r \Phi^s - c_2 k_2^m \Phi^m).
\]

Next we show that there exist positive constants \( k_1, k_2 \leq k \) such that
\[
-b_1 k_1^r \Phi^r + c_1 k_1^l \Phi^l \geq 0, \quad b_2 k_2^r \Phi^s - c_2 k_2^m \Phi^m \geq 0.
\]

In view of \( ls = rm \), it is equivalent to show that for \( \Phi > 0 \)
\[
\frac{b_1^r \Phi^r}{c_1^l \Phi^l} \leq \frac{k_1^r}{k_1^l} \leq \frac{b_2^r \Phi^s}{c_2^m \Phi^m}. \quad (19)
\]

Since \( b_1^r c_1^l \leq b_2^r c_2^m \), it is clear that we can find suitable \( k_1, k_2 \leq k \) such that (19) holds. Note that \( \min\{a_1, a_2\} > \lambda_1 \); thus

\[
\begin{cases}
\begin{aligned}
u_t &\leq u^p \Delta u + u^{p+1}(a_1 - b_1 u^r + c_1 \Phi^l) &\text{in} \ \Omega \times (0, T), \\
\nu_t &\leq v^q \Delta u + v^{q+1}(a_2 + b_2 u^s - c_2 \Phi^m) &\text{in} \ \Omega \times (0, T), \\
u(x, t) &\leq \Phi(x) &\text{on} \ \partial \Omega \times (0, T), \\
u(x, 0) &= k_1 \Phi(x) &\leq u_0(x), \quad \nu(x, 0) = k_2 \Phi(x) &\leq v_0(x) &\text{for} \ x \in \bar{\Omega}.
\end{aligned}
\end{cases}
\]

It follows from Comparison Principle 1 that \((u, v) \geq (k_1 \Phi(x), k_2 \Phi(x))\). This ends the proof. \( \square \)
Lemma 18. Assume that $0 < \alpha < 1$ and $d, \delta > 0$ and that $w$ is a classical solution of
\[
\begin{align*}
  w_t &= dw^\alpha(\Delta w + aw) \quad \text{in } \Omega \times (0, T), \\
  w(x, 0) &= \delta \quad \text{for } x \in \Omega, \\
  w(x, t) &= \delta \quad \text{on } \partial \Omega \times (0, T).
\end{align*}
\]
If $a > \lambda_1$, then $w(x, t)$ blows up in finite time.

Proof. See Lemma 3.1 in [10]. \qed

Proof (Proof of Theorem 2(2)). In view of $\min\{a_1, a_2\} > \lambda_1$, we can choose a smooth subdomain $\Omega_e \subset \Omega$ such that $\lambda_1 < \lambda_1^e \leq \min\{a_1, a_2\}$, where $\lambda_1^e$ is the first eigenvalue of $-\Delta$ in $\Omega_e$ with homogeneous Dirichlet boundary condition.

Define $\delta = \min\{k_1 \min_{\partial \Omega_e} \Phi(x), k_2 \min_{\partial \Omega_e} \Phi(x)\}$; then $\delta > 0$ and we consider the following problem:
\[
\begin{align*}
  u_t &= u^p \Delta u + u^{p+1}(a_1 - b_1 u^r + c_1 v^l) \quad \text{in } \Omega_e \times (0, T^*), \\
  v_t &= v^q \Delta v + v^{q+1}(a_2 - b_2 v^s - c_2 u^{m}) \quad \text{in } \Omega_e \times (0, T^*), \\
  u(x, t) &= \delta \quad \text{for } x \in \Omega_e, \\
  u(x, 0) &= \delta \quad \text{for } x \in \Omega_e.
\end{align*}
\]
(20)

Similarly, we can show $(u, v) \geq (\bar{u}, \bar{v}) \geq (\delta, \delta)$ on $\bar{\Omega}_e \times [0, T)$. Let $(\bar{u}, \bar{v}) = (l_1 w, l_2 w)$ where $l_1, l_2$ are some positive constants to be chosen later and $w$ is a nonnegative function which will be defined later. We show that we can choose $l_1, l_2$ and $w$ so that $(\bar{u}, \bar{v})$ is a subsolution of (20); moreover, $w$ blows up in finite time.

Clearly $(l_1 w, l_2 w)$ is a subsolution of (20) if
\[
\begin{align*}
  w_t &\leq l_1^p \bar{W}^p(\Delta w + w(a_1 - b_1 l_1^r w^r + c_1 l_1^l w^l)) \quad \text{in } \Omega_e \times (0, T^*), \\
  w_t &\leq l_2^q \bar{W}^q(\Delta w + w(a_2 + 2 b_2 l_2^s w^s - c_2 l_2^m w^m)) \quad \text{in } \Omega_e \times (0, T^*), \\
  l_1 w(x, t) &\leq \delta, \quad l_2 w(x, t) \leq \delta \quad \text{on } \partial \Omega_e \times (0, T^*), \\
  l_1 w(x, 0) &\leq \delta, \quad l_2 w(x, 0) \leq \delta \quad \text{for } x \in \Omega_e.
\end{align*}
\]
(21)

As before, we can choose $l_1, l_2$ such that
\[-b_1 l_1^r w^r + c_1 l_1^l w^l \geq 0, \quad b_2 l_2^s w^s - c_2 l_2^m w^m \geq 0.
\]

Thus if we let $a = \min\{a_1, a_2\}$ and $d = \min\{l_1^p \delta^{p-\alpha}, l_2^q \delta^{q-\alpha}\}$ for some constant $\alpha$ and $w$ is the classical solution of
\[
\begin{align*}
  w_t &= dw^\alpha(\Delta w + aw) \quad \text{in } \Omega_e \times (0, T), \\
  w(x, 0) &= \delta \quad \text{for } x \in \Omega_e, \\
  w(x, t) &= \delta \quad \text{on } \partial \Omega_e \times (0, T).
\end{align*}
\]
then $(l_1 w, l_2 w)$ is a subsolution of (20). In view of Lemma 18, $(\bar{u}, \bar{v})$ blows up in finite time in $\bar{\Omega}_e$; the conclusion of the theorem follows. \qed

Proof (Proof of Theorem 3). Let $l s = r_0 m$ for some constant $r_0 > r$. Any nonnegative solution of (1) is clearly a supersolution of the following system:
\[
\begin{align*}
  u_{l_t} &= u_l^p \Delta u_l + u_l^{p+1}(a_1 - b_1 u_l^r + c_1 v_l^l) \quad \text{in } \Omega \times \mathbb{R}^+, \\
  v_{l_t} &= v_l^q \Delta v_l + v_l^{q+1}(a_2 + 2 b_2 u_l^s - c_2 v_l^m) \quad \text{in } \Omega \times \mathbb{R}^+, \\
  u_l(x, t) &= v_l(x, t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
  u_l(x, 0) &= u_0(x), \quad v_l(x, 0) = v_0(x) \quad \text{for } x \in \Omega.
\end{align*}
\]
(22)

For system (22) we can apply the same argument as in the proof of Theorem 2(2) and we conclude that the solution blows up for $b_1^c c_1^2 \leq b_2^c c_1^2$ and $\min\{a_1 - b_1, a_2\} > \lambda_1$. Similarly, if $\min\{a_1, a_2 - b_2\} > \lambda_1$, we reach the same conclusion by modifying the second equation in the system. \qed
References