

# Recent Progress in Large Solutions to Elliptic Problems

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# Outline

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## 1 Statement of the Problem

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## 2 Motivation

- Previous Work
- Assumptions

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- 1 Statement of the Problem
- 2 Motivation
  - Previous Work
  - Assumptions
- 3 Our Results/Contribution
  - Main Results
  - Basic Ideas for Proofs
  - Blowup Rate and Uniqueness

# Statement of the Problem

Existence, Uniqueness and Asymptotic Behavior:

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^N$ ,  $\lambda \in \mathbb{R}$ , and  $b(x) \in C^\alpha(\bar{\Omega}, \mathbb{R}^+)$  for some  $\alpha \in (0, 1)$ ,  $\mathbb{R}^+ := [0, +\infty)$ . A solution of (1) is called large (or explosive) solution, by which we mean a function  $u \in C^2(\Omega)$  such that

$$u(x) \rightarrow +\infty \text{ as } d(x) := \text{dist}(x, \partial\Omega) \rightarrow 0^+.$$



## Previous Results

### The singular boundary problem

$$\begin{cases} -\Delta u = \beta(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \quad (2)$$

- 1916, Bieberbach: for  $\beta(u) = e^u$  on a bounded domain in  $\mathbb{R}^2$ , there exists a unique solution  $u \in C^2(\Omega)$ . Moreover,

$$|u(x) - \sqrt{2} \ln(\text{dist}(x, \partial\Omega)^{-2})| \leq C, \text{ as } \text{dist}(x, \partial\Omega) \rightarrow 0.$$

- 1943, H. Rademacher: extend this result to smooth bounded domains in  $\mathbb{R}^3$ .

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- Does a blow-up solution exist?
- Is such a solution unique, or are there many?
- What is the exact boundary behaviour of blow-up solutions, given as a difference or ratio of some known function of  $d(x)$ .

## Previous Results

1957, Keller  $\Delta u = f(u)$ : Existence of large solution  $\Leftrightarrow$

$$\int_1^{\infty} \frac{dt}{\sqrt{F(t)}} < \infty \text{ where } F(t) = \int_0^t f(s) ds$$

provided that

$$f \in C^1[0, \infty), f(0) = 0, f(s) > 0 \text{ for } s > 0 \text{ and } f'(s) \geq 0 \text{ for } s \geq 0.$$

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- 2000-2007, Huang, Delgado, Feng, López-Gómez, Cîrstea, Rădulescu:  $-\Delta u = \lambda g(u) - b(x)f(u)$ :  $g(u) = u^q$ ,  $b(x) = C_0 d^\nu(x) + o(d(x))$ ,  $f(u) = u^p$ .

## Previous Results on $-\Delta u = a(x)u^{1/m} - b(x)f(u)$

When  $m = 1$ , the existence, uniqueness and blow-up rate has been studied by:

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- Feng,  $f(u) = u^{p/m}$   $p > m$ , JMAA(2007).

## Previous Results on Elliptic Equations

We consider the singular boundary value problem

$$\begin{cases} -\Delta w^m = \lambda w - a(x)w^p & \text{in } \Omega, \\ w = \infty & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 1$ , with boundary  $\partial\Omega$  of class  $C^2$ ,  $\lambda \in \mathbb{R}$ ,  $m > 1$ ,  $p > 1$  and  $a(x) \in C^\alpha(\bar{\Omega}, \mathbb{R}^+)$  for some  $\alpha \in (0, 1)$ ,  $\mathbb{R}^+ := [0, +\infty)$ .

## Theorem (P.Feng, JMAA, 329, 2007)

Consider the radially symmetric semilinear elliptic equation

$$\begin{cases} -\Delta u = \lambda u^{\frac{1}{m}} - a(x)u^{\frac{p}{m}} & \text{in } B_R(0), \\ u = \infty & \text{on } \partial B_R(0), \end{cases} \quad (4)$$

$p > m \geq 1$ ,  $\lambda \in \mathbb{R}$ ,  $a \in C([0, R]; [0, \infty))$  satisfying

$$a > 0 \text{ in } [0, R), \quad \frac{A(r)}{a(r)} \in C^1([0, R]),$$

$$\lim_{r \rightarrow R} \frac{A(r)}{a(r)} = 0 \text{ where } A(r) := \int_r^R a(s) ds.$$

## Theorem (P.Feng, JMAA, 329, 2007)

Then the problem (4) admits a solution  $u$  satisfying

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{M \left( \int_r^R A(s) ds \right)^{-\alpha}} = 1$$

where  $d(x) := \text{dist}(x, \partial B_R(0))$  and

$$M = [\alpha(\alpha + 1)A_0 - \alpha]^\alpha, \quad \alpha = \frac{m}{p - m}.$$

Here

$$A_0 = \lim_{r \rightarrow R} \frac{A(r)^2}{a(r) \int_r^R A(s) ds}.$$

Furthermore, if  $\lambda \geq 0$ , then (4) admits a unique solution.

## Additional Results on Elliptic Systems

(2003, ADE) Jorge Garcia-Melian and Antonio Suarez (Cooperative Lotka-Volterra):

$$\begin{cases} -\Delta u = \lambda u - u^2 + buv & \text{in } \Omega, \\ -\Delta v = \mu v - v^2 + cuv & \text{in } \Omega, \\ u = v = +\infty & \text{on } \partial\Omega \end{cases} \quad (5)$$

(2006, JDE) J. Garcia-Melian, R. Letelier and J. Sabina de Lis (Competitive System):

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q & \text{in } \Omega, \\ u = v = +\infty & \text{on } \partial\Omega \end{cases} \quad (6)$$

## Additional Results on Elliptic Systems

(2006, SIAM JMA) J.I. Diaz, M. Lazzo and P.G. Schmidt (Fluid Dynamics):

$$\begin{cases} \Delta v = \phi, \\ \Delta \phi = |\nabla v|^2 \end{cases} \quad (7)$$

(2007, JDE) H.L. Li and M.X. Wang

$$\begin{cases} -\Delta u = u(a_1 - b_1 u^m - c_1 v^n) & \text{in } \Omega, \\ -\Delta v = v(a_2 - b_2 v^p - c_2 v^q) & \text{in } \Omega, \\ u = v = +\infty & \text{on } \partial\Omega \end{cases} \quad (8)$$

## Additional Results on Elliptic Systems

(2010) P. Feng and J. Ali:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u| = m(|x|)f_1(v)f_2(u), & x \in \mathbb{R}^N \\ \operatorname{div}(|\nabla v|^{q-2}\nabla v) + |\nabla v| = n(|x|)g_1(v)g_2(u), & x \in \mathbb{R}^N \end{cases} \quad (9)$$

$$M(r) = \int_0^r \left[ e^{-t} t^{1-N} \int_0^t e^s s^{N-1} m(s) ds \right]^{1/(p-1)} dt \quad (10)$$

$$N(r) = \int_0^r \left[ e^{-t} t^{1-N} \int_0^t e^s s^{N-1} n(s) ds \right]^{1/(q-1)} dt \quad (11)$$

$$F(r) = \int_0^r \frac{1}{[f_1(\tau)f_2(\tau) + g_1(\tau)g_2(\tau)]^{1/(\nu-1)}} d\tau, \quad (12)$$

where  $\nu := \min\{p, q\}$

## Theorem

*If  $F(\infty) = \infty$ , then system (9) has at least one positive radial solution  $(u(r), v(r)) \in C^2([0, \infty))$ . If  $\lim_{r \rightarrow \infty} M(r) < \infty$  and  $\lim_{r \rightarrow \infty} N(r) < \infty$ , the solutions are bounded. If  $\lim_{r \rightarrow \infty} M(r) = \infty$  and  $\lim_{r \rightarrow \infty} N(r) = \infty$ , there are infinitely many positive entire large solutions.*

## Assumptions on Our Problem

**B1**  $b(x) = 0$  on  $\partial\Omega$  and there exists a positive increasing function  $h \in C^1(0, \delta_0)$  for some  $\delta_0 > 0$  such that

$$\lim_{d(x) \rightarrow 0^+} \frac{b(x)}{h^2(d(x))} = c_0 > 0,$$

and

$$\lim_{d \rightarrow 0^+} \frac{\int_0^d h(s) ds}{h(d)} = 0, \quad \lim_{d \rightarrow 0^+} \left( \frac{\int_0^d h(s) ds}{h(d)} \right)' = l_1 > 0.$$

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**F3**  $f$  is regularly varying at infinity with index  $p > 1$ .

## More Assumptions.

and the following assumptions on  $g(t) \in C^1[0, +\infty)$

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**G2**  $g(t)/t$  is nonincreasing on  $(0, +\infty)$ .

**G3**  $g(t)$  is regularly varying at infinity with index  $0 < q < 1$ .

# Regular Variation

## Definition

A positive measurable function  $f$  defined on  $[a, \infty)$  for some  $a > 0$  is called regularly varying at infinity with index  $p$  if for all  $\xi > 0$ ,

$$\lim_{t \rightarrow \infty} f(\xi t)/f(t) = \xi^p.$$

## Example

$t^p$ ,  $t^p \ln(1 + t)$ ,  $(t \ln(1 + t))^p$  are regularly varying with index  $p$ ; but not  $2 + \sin t$ .

## Main Result

## Theorem (P.Feng, JMAA, 356, 2009)

Suppose  $f$  and  $g$  satisfy (F1-3), (G1-3) and  $b(x)$  satisfies B1. Then for any  $\lambda \in \mathbb{R}$ , equation (1) admits a unique large solution  $u$ . Moreover, we have

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{Z(d(x))} = M, \quad M := \left[ \frac{2 + l_1(p-1)}{c_0(p+1)} \right]^{1/(p-1)},$$

and the function  $Z(t)$  is defined through

$$\int_{Z(t)}^{\infty} \frac{1}{\sqrt{2F(s)}} ds = \int_0^t h(s) ds, \quad t \in (0, \delta_0).$$

# A Corollary

## Corollary

Let  $f(u) = u^p$ ,  $p > 1$ ,  $g(u) = u^q$ ,  $0 < q < 1$ , and  $b(x) \sim c_0(d^\nu(x))$ , then  $h(d) = d^{\nu/2}(x)$ ,  $l_1 = \frac{2}{\nu+2}$ . Any solution  $u$  to (1) satisfies

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{d^{-\alpha}(x)} = M$$

where  $\alpha = \frac{\nu+2}{p-1}$ ,  $M = \left[ \frac{\alpha(\alpha+1)}{c_0} \right]^{\frac{1}{p-1}}$ .

# Preliminary Results

## Lemma

Assume  $f$  satisfies (F1-2), then the following are equivalent:

(i)  $f \in \mathbb{R}_p$ ; (ii)  $\lim_{t \rightarrow \infty} \frac{tf'(t)}{f(t)} = p$ ; (iii)  $\lim_{t \rightarrow \infty} \left( \frac{F(t)}{f(t)} \right)' = 1/(1+p)$ .

## Lemma

If  $f(t)$  satisfies (F1-3), then  $Z(t)$  in the main theorem satisfies the following properties:

$$(1) \lim_{t \rightarrow 0^+} \frac{Z(t)}{Z''(t)} = \lim_{t \rightarrow 0^+} \frac{Z(t)}{Z'(t)} = \lim_{t \rightarrow 0^+} \frac{Z'(t)}{Z''(t)} = 0;$$

$$(2) \lim_{t \rightarrow 0^+} \frac{Z''(t)}{h^2(t)f(\xi Z)} = \frac{1}{\xi^p} \frac{pl_1 + 2 - l_1}{p+1} \text{ for any } \xi > 0;$$

$$(3) \lim_{t \rightarrow 0^+} \frac{g(\xi Z(t))}{Z''(t)} = 0 \text{ for any } \xi > 0.$$

# Comparison Principles

## Lemma

Let  $\Omega_0$  be a smooth bounded domain in  $\mathbb{R}^N$ . Assume  $f(u)$  satisfies F(1-2) and  $g(u)$  satisfies G(1-2),  $b(x)$ ,  $r(x)$  are  $C^\alpha$  functions on  $\bar{\Omega}_0$  such that  $r(x) \geq 0$ ,  $b(x) > 0$  on  $\Omega_0$  and  $\lambda \in \mathbb{R}$ . Let  $u_1, u_2 \in C^2(\Omega_0)$  be positive functions such that

$$-\Delta u_1 - \lambda g(u_1) + b(x)f(u_1) - r(x) \geq 0 \geq -\Delta u_2 - \lambda g(u_2) + b(x)f(u_2) - r(x) \text{ in } \Omega_0 \quad (13)$$

and

$$\liminf_{d(x) \rightarrow 0} (u_1(x) - u_2(x)) \geq 0,$$

where  $d(x) := \text{dist}(x, \partial\Omega_0)$ . Then  $u_1 \geq u_2$  in  $\Omega_0$ .

## Proof of Comparison Principle

It follows from (13) that for any nonnegative function  $\Phi \in H^1(\Omega_0)$  with compact support, we have

$$\begin{aligned} & \int_{\Omega_0} \nabla u_1 \nabla \Phi - \lambda g(u_1)\Phi + b(x)f(u_1)\Phi - r\Phi \\ \geq & \int_{\Omega_0} \nabla u_2 \nabla \Phi - \lambda g(u_2)\Phi + b(x)f(u_2)\Phi - r\Phi \end{aligned} \quad (14)$$

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Let  $\epsilon_1 > \epsilon_2 > 0$  and denote

$$\Omega_+(\epsilon_1, \epsilon_2) = \{x \in \bar{\Omega}_0 : u_2(x) + \epsilon_2 > u_1(x) + \epsilon_1\},$$

and

$$v_i = (u_i + \epsilon_i)^{-1} [(u_2(x) + \epsilon_2)^2 - (u_1(x) + \epsilon_1)^2]^+.$$

## Proof of Comparison Principle

It follows from (13) that for any nonnegative function  $\Phi \in H^1(\Omega_0)$  with compact support, we have

$$\begin{aligned} & \int_{\Omega_0} \nabla u_1 \nabla \Phi - \lambda g(u_1)\Phi + b(x)f(u_1)\Phi - r\Phi \\ & \geq \int_{\Omega_0} \nabla u_2 \nabla \Phi - \lambda g(u_2)\Phi + b(x)f(u_2)\Phi - r\Phi \end{aligned} \quad (14)$$

Let  $\epsilon_1 > \epsilon_2 > 0$  and denote

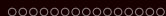
$$\Omega_+(\epsilon_1, \epsilon_2) = \{x \in \bar{\Omega}_0 : u_2(x) + \epsilon_2 > u_1(x) + \epsilon_1\},$$

and

$$v_i = (u_i + \epsilon_i)^{-1} [(u_2(x) + \epsilon_2)^2 - (u_1(x) + \epsilon_1)^2]^+.$$

Notice that  $v_i \in H^1(\Omega_0)$  and it has compact support in  $\Omega_0$  (it vanishes outside  $\Omega_+$ ).





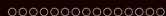
Replacing  $\Phi$  in (14) by  $v_1, v_2$  and applying integration by parts and subtraction yields

$$\begin{aligned} & \int_{\Omega_+} (\nabla u_1 \nabla v_1 - \nabla u_2 \nabla v_2) - \int_{\Omega_+} \lambda(g(u_1)v_1 - g(u_2)v_2) \\ & \geq \int_{\Omega_+} b(x)(f(u_2)v_2 - f(u_1)v_1) + \int_{\Omega_+} r(x)(v_1 - v_2) \end{aligned} \quad (15)$$

A simple calculation shows that the first integral on the left-hand side of (15) equals

$$- \int_{\Omega_+} \left( \left| \nabla u_2 - \frac{u_2 + \epsilon_2}{u_1 + \epsilon_1} \nabla u_1 \right|^2 + \left| \nabla u_1 - \frac{u_1 + \epsilon_1}{u_2 + \epsilon_2} \nabla u_2 \right|^2 \right) dx \leq 0.$$





As  $0 < \epsilon_2 < \epsilon_1 \rightarrow 0$ , the second term on the left-hand side of (15) equals

$$- \int_{\Omega_+(0,0)} \lambda [g(u_1)/u_1 - g(u_2)/u_2] [u_2^2 - u_1^2]^+ dx \leq 0.$$

On the other hand, the first term on the right-hand side converges to

$$\int_{\Omega_+(0,0)} b(x) \left( \frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right) (u_2^2 - u_1^2) dx > 0$$



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while the second term converges to

$$\int_{\Omega_+(0,0)} r(x) (1/u_1 - 1/u_2) (u_2^2 - u_1^2) dx \geq 0.$$

Therefore, we should have a contradiction unless  $\Omega_+(0,0)$  has measure 0, i.e.,  $u_1 \geq u_2$  on  $\Omega_0$ .

# Uniform Boundness

## Lemma

Let  $B(R) \subset \mathbb{R}^n$  be an arbitrary ball centered at  $x_0$  and consider the auxiliary problem

$$\begin{cases} -\Delta u = \lambda g(u) - Af(u) & \text{in } B, \\ u = \tau & \text{on } \partial B, \end{cases} \quad (16)$$

where  $\lambda \in \mathbb{R}$ ,  $A > 0$  and  $\tau > t_0$  where  $t_0$  is the unique zero of  $\psi(t) = Af(t) - \beta g(t)$  as defined before. Then there exists a constant  $M := M(R)$  such that any solution  $u_\tau$  of (16) satisfies  $\|u_\tau\|_{C(B)} \leq M$ .

# Proof of the Existence

Consider the following perturbed problem

$$\begin{cases} -\Delta u = \lambda g(u) - (b(x) + \frac{1}{n^\gamma})f(u), & x \in \Omega, \\ u = n, & x \in \partial\Omega, \end{cases} \quad (17)$$

where  $\gamma > 0$  satisfies  $p > q + \gamma$  where  $p$  and  $q$  are defined as in F3 and G3, respectively. Since 0 is a subsolution and  $n$  is a supersolution for  $n$  sufficiently large, (17) admits a solution  $u_n \in C^{2,\alpha}(\bar{\Omega})$  with  $u_n \leq n$ . Moreover, Lemma 10 shows that  $\{u_n\}_n$  is increasing. Our purpose is to pass to the limit as  $n \rightarrow \infty$ . Thanks to Lemma 11,  $u_n$  is uniformly bounded on every compact subdomain of  $\Omega$ . By the monotonicity of  $\{u_n\}$ , we conclude  $u_n \rightarrow \underline{u}$  in  $L_{loc}^\infty(\Omega)$ . Finally, standard elliptic regularity arguments lead to  $u_n \rightarrow \underline{u}$  in  $C_{loc}^{2,\alpha}(\Omega)$ .

# Sub- and Supersolution Method

## Lemma

Suppose  $\underline{u}$  and  $\bar{u}$  satisfy

$$-\Delta \underline{u} \leq \lambda g(\underline{u}) - b(x)f(\underline{u}) \text{ in } \Omega,$$

$$-\Delta \bar{u} \geq \lambda g(\bar{u}) - b(x)f(\bar{u}) \text{ in } \Omega,$$

$\lim_{d(x) \rightarrow 0^+} \underline{u}(x) = \lim_{d(x) \rightarrow 0^+} \bar{u}(x) = \infty$  and  $\underline{u} \leq \bar{u}$  in  $\Omega$ . Then (1) admits a solution  $u \in C^2(\Omega)$  satisfying  $\underline{u} \leq u \leq \bar{u}$  in  $\Omega$ .

# Exact Blowup Rate

Let Define

$$\xi_2 = \left[ \frac{2 + l_1(p-1)}{(c_0 - 2\epsilon)(p+1)} \right]^{\frac{1}{p-1}}, \quad \xi_1 = \left[ \frac{2 + l_1(p-1)}{(c_0 + 2\epsilon)(p+1)} \right]^{\frac{1}{p-1}}.$$

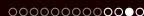
Let  $\mu \in (0, \delta)$  be arbitrary. We define

$$\bar{u}_\mu = \xi_2 Z(d(x) - \mu), \quad x \in \Omega_{2\delta} \setminus \Omega_\mu,$$

and

$$\underline{u}_\mu = \xi_1 Z(d(x) + \mu), \quad x \in \Omega_{2\delta - \mu}.$$

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\} \text{ and } \partial\Omega_\delta := \{x \in \Omega : d(x) = \delta\}$$



## Exact Blowup Rate

Let  $w$  be an arbitrary solution of

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u), & x \in \Omega \setminus \bar{\Omega}_\delta, \\ u = 1, & x \in \partial\Omega, \\ u = +\infty, & x \in \partial\Omega_\delta. \end{cases}$$

Passing the limit  $\mu \rightarrow 0^+$ , we see that

$$\xi_1 Z(d(x)) \leq u(x) + w(x) \leq \xi_2 Z(d(x)) + 2w(x), \quad x \in \Omega_\delta,$$

which implies

$$\xi_1 \leq \liminf_{d(x) \rightarrow 0^+} \frac{u(x)}{Z(d(x))} \leq \limsup_{d(x) \rightarrow 0^+} \frac{u(x)}{Z(d(x))} \leq \xi_2.$$

Finally, we set  $\epsilon \rightarrow 0$  to obtain the exact blow-up rate.

## Proof of Uniqueness

$\lim_{d(x) \rightarrow 0} u_1(x)/u_2(x) = 1$ . Hence, for any  $\epsilon \in (0, 1)$ , there exists  $\delta > 0$  which depends on  $\epsilon$  such that

$$(1 - \epsilon)u_2 \leq u_1 \leq (1 + \epsilon)u_2, \quad x \in \Omega_\delta.$$

Clearly,  $u_1$  is a positive solution of

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u), & x \in \Omega_\delta, \\ u = u_1, & x \in \partial(\Omega \setminus \Omega_\delta). \end{cases} \quad (18)$$

Thus for  $x \in \Omega \setminus \Omega_\delta$ , we have

$$(1 - \epsilon)u_2 \leq u_1 \leq (1 + \epsilon)u_2.$$

Letting  $\epsilon \rightarrow 0$ , we see that  $u_1 \equiv u_2$  in  $\Omega$ .

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- The exact blow-up rate enables us to establish the uniqueness.

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